~~Appendix I~~

THE STATISTICS OF GRAVITATIONAL CLUSTERING*

by

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March 1965

*Submitted to Harvard University in partial fulfillment of the requirements for the Ph.D.

*Supported in part by NASA Research Grant NsG 540-22/05-002.

ACKNOWLEDGEMENT

I would like to express my sincere thanks to Prof. David Layzer for introducing me to the study of cosmology and for the valuable help, stimulation, and encouragement received from him during the progress of this work.

ABSTRACT

The research described in this thesis consists of an application of the hierarchical distribution function formalism to the cosmogonical problem of the development of irregularities in an expanding universe.

The cosmological setting of the work to follow is first presented in a brief resumé of certain aspects of Layzer's hypothesis of gravitational clustering. Next, the equations of motion of a cosmic distribution of mutually gravitating point masses are obtained in a particularly convenient set of position, velocity, and time coordinates. In these coordinates the gross expansion of the universe is transformed away and in its place an apparent background of negative mass and an explicit time dependence of the gravitational constant appear.

The formalism of distribution and correlation functions is then developed with emphasis placed upon the generating functional of Bogolioubov. The generating functional method is employed in a new derivation of the equations of motion for the correlation functions. Layzer's clustering spectrum is redefined in terms of the two-particle correlation function to obtain a quantity more appropriate to a

particulate distribution.

Various models of clustering are next studied and the simplicity of the correlation functions as opposed to the distribution functions exhibited. The clustering spectrum for a three-level hierarchical distribution of mass points is explicitly calculated with the aid of the generating functional. It is found that pronounced clustering on a distance scale λ does not necessarily imply a peak in the clustering spectrum at $k \approx 1/\lambda$ as one might intuitively expect.

The dynamics of clustering are then investigated and the energy theorem for the peculiar kinetic and potential energies derived. The simplification resulting from the assumption of weak clustering is exploited in the derivation of a useful integral equation for the clustering spectrum. This integral equation is numerically solved under the assumption of certain simple initial conditions. As expected, it is found that clustering on a distance scale much smaller than Jeans' critical wavelength is overcome by Landau damping or phase mixing but that clustering on a much larger scale proceeds to grow.

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CHAPTER I

INTRODUCTION

The work to be described in this thesis was motivated by certain cosmological speculations of D. Layzer. It represents an effort to provide a suitable mathematical framework for the examination and extension of these ideas. It is therefore appropriate to briefly outline the relevant hypotheses in order to make clear the cosmological setting of the work to follow.

I. THE COSMOLOGICAL SETTING

Most modern cosmogonical theories are based upon the assumption that more or less structureless, perhaps turbulent, clouds of dust and gas can condense or fragment to form stars and planets. Layzer (1) has investigated this process of fragmentation and has concluded that it cannot occur under circumstances likely to be realized in nature. To circumvent this difficulty Layzer (2) had earlier advanced the hypothesis of gravitational clustering. He stated the hypothesis as follows:

> Consider a cosmic distribution of matter in which there are slight local irregularities. As the universe expands the irregularities become more and more pronounced until finally self-gravitating systems separate out. The newly formed systems play the role of particles in a new cosmic distribution, which will also have slight local irregularities, in general, and the stage

is set for a repetition of the clustering process. Specifically, the planets and stars are assumed to form near the beginning of the expansion by gravitational clustering of gas and dust. Then multiple stellar systems, star clusters, and galaxies are assumed to form in a cosmic distribution of stars. And finally multiple galactic systems, groups of galaxies, and great clusters of galaxies are assumed to form by gravitational clustering in a cosmic distribution of galaxies.

Layzer recognizes that at each stage of the clustering process there must exist some organization at all levels of astronomical size. For example, even as stars and planets are being formed there must already be incipient clustering on the galactic and multi-galactic levels. The principal hypothesis is that significant small-scale clustering precedes significant large scale clustering.

In order to provide a measure of the degree of clustering associated with systems of each characteristic size Layzer introduces the clustering spectrum, the three-dimensional Fourier transform of the autocorrelation function of the peculiar part of the cosmic mass distribution. Let the mass density of the cosmic distribution be divided into a part independent of position and a part of vanishing space average.

$$\rho(\vec{x}) = \bar{\rho} + \tilde{\rho}(\vec{x}) \tag{1}$$

where

$$\langle \tilde{\rho}(\vec{x}) \rangle = 0 \tag{2}$$

The cosmological principle is assumed to hold and therefore the distribution is, on some sufficiently large scale, statistically homogeneous and isotropic. This implies

$$\langle \tilde{\rho}(\vec{x}+\vec{x}')\tilde{\rho}(\vec{x}')\rangle \equiv \lim_{\Omega\to\infty}\frac{1}{\Omega}\int \tilde{\rho}(\vec{x}+\vec{x}')\tilde{\rho}(\vec{x}')\,d\vec{x}' = \langle \tilde{\rho}^2\rangle\, f(x) \tag{3}$$

where f(x), the autocorrelation function, depends on the magnitude of $\vec{x}$ only. Note that the limit is assumed to be independent of the location of the volume Ω.

Consider the gravitational energy per unit mass possessed by this cosmic distribution. It is convenient to think of it as the limit of the specific potential energy of a finite distribution as the size of the distribution tends toward infinity.

$$U_m = \lim_{\Omega\to\infty}\frac{-G}{2\bar{\rho}\Omega}\int_\Omega d\vec{x}\int_\Omega d\vec{x}'\,\frac{(\bar{\rho}+\tilde{\rho}(\vec{x}))(\bar{\rho}+\tilde{\rho}(\vec{x}'))}{|\vec{x}-\vec{x}'|}$$

$$= \lim_{\Omega\to\infty} -\left[\frac{G\bar{\rho}}{2\Omega}\int_\Omega d\vec{x}\int_\Omega d\vec{x}'\,\frac{1}{|\vec{x}-\vec{x}'|} + \frac{G}{\Omega}\int_\Omega d\vec{x}\int_\Omega d\vec{x}'\,\frac{\tilde{\rho}(\vec{x})}{|\vec{x}-\vec{x}'|} + \frac{G}{2\bar{\rho}\Omega}\int_\Omega d\vec{x}\int_\Omega d\vec{x}'\,\frac{\tilde{\rho}(\vec{x})\,\tilde{\rho}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right] \tag{4}$$

The limit of the first term does not exist, but as this term is independent of the clustering we may regard it as an uninteresting although infinite constant. In the

second and third terms make the change of variables $\vec{x} \to \vec{x}+\vec{x}'$, and let the x integration extend over all space. We then find

$$\lim_{\Omega\to\infty} \frac{G}{\Omega} \int_\Omega d\vec{x} \int_\Omega d\vec{x}' \frac{\tilde{\rho}(\vec{x})}{|\vec{x}-\vec{x}'|}$$

$$= G \int d\vec{x} \frac{1}{x} \lim_{\Omega\to\infty} \frac{1}{\Omega} \int d\vec{x}' \, \tilde{\rho}(\vec{x}+\vec{x}') \qquad (5)$$

$$= 0$$

And similarly,

$$\lim_{\Omega\to\infty} \frac{-G}{2\bar{\rho}\Omega} \int_\Omega d\vec{x} \int_\Omega d\vec{x}' \frac{\tilde{\rho}(\vec{x})\,\tilde{\rho}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$= \frac{-G}{\bar{\rho}} \int d\vec{x} \frac{1}{x} \lim_{\Omega\to\infty} \frac{1}{\Omega} \int_\Omega \tilde{\rho}(\vec{x}'+\vec{x})\,\tilde{\rho}(\vec{x}')\, d\vec{x}' \qquad (6)$$

$$= \frac{-G\langle \tilde{\rho}^2 \rangle}{2\bar{\rho}} \int \rho(x) \frac{1}{x} \, d\vec{x}$$

Thus, the contribution of the density fluctuations to the specific potential energy may be written

$$U_m^1 = \int \frac{\rho(x)}{x} \, d\vec{x} \left[-\frac{G\langle \tilde{\rho}^2 \rangle}{2\bar{\rho}} \right] \qquad (7)$$

It is assumed that f(x) decreases sufficiently rapidly at large distances so that the integral converges. Let us express this integral in terms of the Fourier transform of f(x).

$$\rho(x) = \int e^{2\pi i \vec{k}\cdot\vec{x}} \, \hat{\rho}(k) \, d\vec{k} \qquad (8)$$

Using the well known Fourier transform of 1/x,

$$\frac{1}{x} = \int e^{2\pi i \vec{\xi}\cdot\vec{r}} \frac{1}{\pi \xi^2} d\vec{\xi} \tag{9}$$

and Parseval's theorem in the form,

$$\int g(\vec{r}) h(\vec{r}) d\vec{r} = \int \bar{g}(\vec{\xi}) \bar{h}(-\vec{\xi}) d\vec{\xi} \tag{10}$$

we obtain finally the following result.

$$\begin{aligned} U_m &= \frac{-G\langle \bar{\rho}^2 \rangle}{\bar{\rho}} \int \frac{1}{\pi \xi^2} \mathcal{S}^2(\xi) d\vec{\xi} \\ &= \frac{-2G\langle \bar{\rho}^2 \rangle}{\bar{\rho}} \int_0^\infty \mathcal{S}^2(\xi) d\xi \end{aligned} \tag{11}$$

$\mathcal{S}^2(k)$, the clustering spectrum, thus describes the distribution of peculiar potential energy in wave number space. Layzer interprets $\mathcal{S}^2(k)$ as indicating the degree of clustering associated with systems of characteristic size 1/k. For that reason the clustering spectrum assumes a central role in his cosmology. He assumes that for a distribution characterized by clustering on some well defined scale, say L, $\mathcal{S}^2(k)$ will exhibit a peak in the neighborhood of k=1/L. The actual situation will become somewhat clearer when the spectrum function is actually calculated for certain specific distributions in chapter IV.

When one observes the distribution of matter in astronomical systems throughout the visible universe one

notices an approximately hierarchical organization. That is, astronomical systems do not occur with equal frequency in all sizes, but rather they seem more or less concentrated at certain levels--stars, stellar clusters, galaxies etc. To be sure, this hierarchical organization is not at all precise, but it does seem to exist. In Layzer's terms the clustering spectrum appears to exhibit peaks. Since it seems unlikely that a well defined hierarchical organization existed very early in the expansion, the development of this organization is one of the outstanding features to be accounted for by any successful cosmology. Layzer conjectures that the spectrum function is unstable in the sense that small positive perturbations in the initial spectrum tend to grow at the expense of negative fluctuations. A qualitative argument in support of this conjecture proceeds as follows.

Imagine a sharply hierarchical cosmic distribution of matter. Let the lowest level consist of pre-stars, relatively compact masses of dust and gas in the process of Kelvin contraction. In this process gravitational energy is converted into heat and is then radiated away. One may view this radiation as either an outflow of positive energy or alternatively as an inflow of binding energy. Now by the

assumption of a hierarchical organization the pre-stars are grouped in clusters, the clusters in superclusters, and so on. Inelastic collisions between pre-stars will tend to transfer orbital kinetic energy of the pre-stars into thermal energy which will then be radiated away. The clusters will thus tend to become increasingly compact. In other terms, the inflow of binding energy into the pre-stars represented by their radiation is fed in turn into the clusters. In a similar way collisions between clusters result in a flow of binding energy from the clusters to the superclusters. One thus obtains a picture of binding energy cascading from small-scale levels to large-scale levels.

Imagine now that some level, the cluster level for example, is particularly well developed. In that case the clusters will be unusually compact and collisions between pre-stars will be frequent. There will consequently be a high rate of binding energy inflow into the cluster level. On the other hand collisions between clusters will be infrequent and the clusters themselves will be relatively resistant to disruption during these encounters. Thus, there will be a low rate of binding energy outflow from the cluster level to the supercluster level. The net result is that organization on the cluster level will become even more intense at the expense of organization on the pre-star and

supercluster levels.

Layzer likens this flow of binding energy to the cascading of sand through a vertical series of funnels, each funnel representing a level of the clustering hierarchy. In this analog the outflow from an overly full funnel tends to diminish because of increased friction between the grains of sand.

Although this argument has been predicated upon a distribution which is already sharply hierarchical, its conclusion , that intense clustering levels tend to develop at the expense of less intensly developed levels, may presumably apply to distributions in which there is only a tendency toward hierarchical organization. If that is the case the spectrum function is unstable in the sense mentioned.

One feature of the purely gravitational analysis to follow is the absence of any characteristic linear dimension. From this it follows that the actual occurence of characteristic clustering scales cannot be determined by the gravitational dynamics alone but rather by the initial conditions. The present work is therefore necessarily incomplete and is intended to complement an investigation of the non-gravita-

tional processes which govern the development of irregularities early in the course of the expansion.

The analysis presented in thesis is limited also in another way. As will be shown in chapter V, the transfer of binding energy from one clustering level to another is essentially a non-linear phenomenon. However, because of the great complexity of a suitable non-linear treatment the analysis presented here is limited to the linearized approximation.

This thesis then does not represent in any sense a complete solution of the clustering problem but it is hoped that it is a step in that direction.

CHAPTER II

THE EQUATIONS OF MOTION

The research to be described in this thesis has been directed at understanding the purely gravitational aspects of cosmic clustering. Thus, only gravitational forces are to be treated and such phenomena as thermal radiation are to be ignored. As a consequence, the details of clustering on the pre-stellar scale will be inaccurate as soon as the pre-stars become dense enough for non-gravitational forces and radiation to become important. It is expected however that the general structure of clustering at large characteristic distances and the details of the flow of peculiar binding energy from one clustering level to another should be contained in this purely gravitational treatment.

I. THE COSMIC DISTRIBUTION

At this point it is necessary to choose a particular representation of the distribution of matter. The two most reasonable representations are a distribution of point particles and a continuous distribution of mass density. Although these two representations appear very different they become much less so when the problem is treated in statistical terms. In that case the probability of finding a

particle at some point in the particulate view behaves very much like the average mass density in the continuous view. The only difference between the two representations arises from the effects of close encounters between particles, essentially relaxation phenomena. One may say that when statistically treated the particulate representation approaches the continuous representation as the average particle mass approaches zero and the number density of particles approaches infinity in such a way as to keep their product constant. We will call this the continuum limit. In this sense the particulate representation is the more general of the two since it contains the continuum representation as a limit.

If one intends to use the particulate description only as an intermediate step and to always take the continuum limit the two views are precisely equivalent. The particulate view is still to be preferred however because it corresponds more closely to the situations considered in the standard treatments of non-equilibrium statistical mechanics. These results may then be applied with very little modification to the present problem.

We therefore choose to represent the cosmic distribution as a uniformly expanding, statistically homogeneous

and isotropic distribution of mutually gravitating point particles. As we shall later see the continuum limit is adequate for the description of clustering at large distances, the only region in which the purely gravitational theory is valid in any case. It will therefore be unnecessary to identify the particles of the distribution with specific physical entities such as molecules or stars. In the final results no reference to the nature of the particles will appear. We may in fact make the assumption that the particles are all of equal mass when the resultant simplification becomes sufficiently profitable.

II. THE CO-MOVING COORDINATE SYSTEM

Consider now the equations of motion of the point particles. Let $\vec{x}_j$ and M_j be the position vector and mass respectively of the jth particle. Let G denote the gravitational constant. The Newtonian equations of motion are then

$$\frac{d^2\vec{x}_j}{dt^2} = -\sum_{k\neq j} M_k G \frac{(\vec{x}_j - \vec{x}_k)}{|\vec{x}_j - \vec{x}_k|^3} \tag{12}$$

Equation (12) is valid of course only in an inertial coordinate system. In such a system the universe is, on the average, in a state of uniform expansion. Since in an expanding universe one and only one point may be at rest with respect to any inertial frame all regions of the

universe are not equivalently treated. It will be convenient to introduce a new set of coordinates which expand with the mean motion of the universe. In this system all points will be on an equal footing.

Let $\vec{x}(t)$ be the inertial coordinates of a representative point of the smoothed out or averaged distribution. This distribution is uniformly expanding and so one may choose an inertial frame whose origin is at rest with respect to the local matter. In that case one has the following relation.

$$\vec{X}(t) = \vec{X}(t_0) R(t) \tag{13}$$

R(t) is called the expansion parameter and is defined to be equal to unity at time t_0, the time at which initial conditions are to be imposed.

We shall now define a new set of particle coordinates by the relation

$$\vec{X}_j = \vec{Y}_j R(t) \tag{14}$$

If the $\vec{y}_j$ are held constant the $\vec{x}_j$ simply follow the universal expansion. The $\vec{y}_j$ are thus co-moving coordinates. Let us also introduce a new time-like variable s defined by the following differential equation and initial condition.

$$\frac{dS}{dt} = \frac{1}{R^2} \tag{15a}$$

and

$$S(t_0) = 0 \tag{15b}$$

Substituting these new variables into equation (12) one obtains the transformed equation of motion.

$$\frac{d^2\vec{Y}_j}{dS^2} = -RG\sum_{k\neq j} \frac{M_k(\vec{Y}_j - \vec{Y}_k)}{|\vec{Y}_j - \vec{Y}_k|^3} - Y_j R\frac{d}{dS}\frac{1}{R^2}\frac{dR}{dS} \tag{16}$$

In order to simplify the second term on the right hand side one must consider the time dependence of R.

Newtonian gravitational theory cannot deal consistently with infinite, homogeneous distributions of matter. One must therefore appeal to general relativity to provide a specification of R(t). The appropriate solutions of the field equations are the Einstein-Friedmann solutions, given for example by Landau and Lifshitz (3).

Consider the solution corresponding to a pressure-free or dustlike distribution of matter. Let D be the distance between two particular dust grains. If this distance is small compared to the average gravitational radius of curvature of the universe it is a consequence of the Einstein-Friedmann solutions that D obeys the equation

$$\frac{d^2 D}{dt^2} = \frac{-4\pi}{3} D G \rho \qquad (17)$$

This is precisely the result obtainable from the Newtonian theory by considering the distribution of dust to be finite and spherical in shape. Neither the radius of the sphere nor the location of its center matters. It is merely necessary that the two dust particles under consideration lie within it. We see then that Newtonian theory is applicable to an infinite distribution if the distribution is regarded as the limit of concentric spherical distributions of equal mass density as their radii tend to infinity.

It has been shown by Irvine (4) that in a universe in which the irregularities are characterized by a scale length small compared with the average gravitational radius of curvature, and in which the fluctuating part of the velocity field is small compared with the speed of light, Newtonian gravitation provides an accurate description of the development of the irregularities. Relativity need be invoked only to specify the correct limiting proceedures for the interpretation of such quantities as the average force on some particle due to all other particles, quantities which are otherwise not well defined.

Returning for the moment to the inertial coordinate

system and ordinary time, let $\vec{x}(t)$ be the inertial coordinates of a point co-moving with the average local distribution of matter. Since the infinite universe may be thought of as a limit of spherical distributions of matter, each concentric with the origin, and each of the same average density, the equation of motion for $\vec{x}(t)$ is simply

$$\frac{d^2\vec{x}(t)}{dt^2} = \frac{-4\pi}{3} x^3(t)\bar{\rho}(t) G \frac{\vec{x}(t)}{x^3(t)} \tag{18}$$

where $\bar{\rho}(t)$ is the average density at time t. Substituting equation (13) we find

$$\frac{d^2R}{dt^2} = -\frac{4\pi}{3} G R(t)\bar{\rho}(t) \tag{19}$$

Now from mass conservation we may conclude

$$\bar{\rho}(t) R^3(t) = \bar{\rho}(t_0) \tag{20}$$

Returning to the variable s and substituting equation (20) into equation (19) we obtain

$$\frac{d}{ds}\frac{1}{R^2}\frac{dR}{ds} = -\frac{4\pi}{3} G \bar{\rho}(s=0) \tag{21}$$

Substituting this result into equation (16) we finally obtain

$$\frac{d^2\vec{Y}_j}{ds^2} = -RG\sum_{k\neq j} M_k \frac{(\vec{Y}_j-\vec{Y}_k)}{|Y_j-Y_k|^3} + RG\frac{4\pi}{3}\bar{\rho}(0)\vec{Y}_j \tag{22}$$

The first term on the right hand side of this equa-

tion is of the form of a sum of Newtonian forces with the peculiar difference that the gravitational constant is multiplied by the expansion parameter R. We may consider the combination R(s)G to be a new gravitational constant which is now an explicit function of the time.

To interpret the second term note that if RG is considered to be the new gravitational constant this term is of the form of the gravitational field interior to a uniform sphere of density $-\bar{\rho}(0)$ centered about the origin. Again considering the universe to be the limit of such spheres we see that this second term represents a negative background force due to a distribution of negative mass equal in magnitude of density to the average density of real mass at time t_0. But in the co-moving system the density of real mass is constant. Thus the negative background mass compensates for the average distribution of real mass indefinitely. This is of course perfectly reasonable. In the co-moving frame we have transformed away the decelerating expansion. Yet the first term on the right hand side of equation (22) is of the same form as the original force. Therefore we must have introduced a negative background to cancel the net attraction of the gravitating particles.

Before leaving this chapter it will be useful to

investigate the behavior of the expansion parameter as a function of s. To do this we integrate equation (21) twice. The first integration yields

$$\frac{1}{R^2}\frac{dR}{ds} = -\frac{4\pi}{3} G \bar{\rho}(0) s + C_1 \tag{23}$$

The second integration yields

$$-\frac{1}{R} = -\frac{4\pi}{3} G \bar{\rho}(0) \frac{s^2}{2} + C_1 s + C_2 \tag{24}$$

Since we require that R(0)=1 it follows that $c_2=-1$. The combination $4\pi G\bar{\rho}(0)$ appears repeatedly in the development to follow. It is therefore convenient to adopt the notation

$$4\pi \bar{\rho}(0) G = \omega^2 = 4\pi n m' G / m \tag{25}$$

This combination of constants plays a role in the gravitational problem analogous to that of the square of the plasma frequency in the theory of plasma waves. This latter quantity is given by

$$4\pi n e^2 / m = \omega_p^2 \tag{26}$$

where n is the number density of electrons and e and m are the electronic charge and mass respectively. It is useful also to express c_1 in terms of a ratio of immediate physical significance.

At the initial time t_0 select some member of the sequence of spherical distributions whose limit is to be taken as the infinite universe. Suppose the radius of that distribution to be a. The distribution has kinetic energy of expansion T as well as gravitational energy V. These quantities are easily found to be

$$\begin{aligned} T &= \int_0^a \frac{\bar{\rho}(0)}{2}\left(x\left.\frac{dR}{dt}\right]_{t=t_0}\right)^2 4\pi x^2 dx \\ &= \int_0^a \frac{\bar{\rho}(0)}{2}\left(\left.\frac{dR}{ds}\right]_{s=0}\right)^2 4\pi x^4 dx \\ &= \int_0^a \frac{\bar{\rho}(0)}{2} c_1^2\, 4\pi x^4 dx \\ &= 2\pi \bar{\rho}(0) c_1^2 \frac{a^5}{5} \end{aligned} \tag{27}$$

and

$$\begin{aligned} V &= -\int_0^a \tfrac{4}{3}\pi x^3 \bar{\rho}(0) G \tfrac{1}{x}\, 4\pi x^2 \bar{\rho}(0)\, dx \\ &= -\frac{16\pi^2}{3}\bar{\rho}(0)^2 G \frac{a^5}{5} \end{aligned} \tag{28}$$

Let α be the ratio of the total energy to the gravitational energy.

$$\begin{aligned} \alpha &\equiv \frac{T+V}{V} \\ &= 1 - \frac{3c_1^2}{8\pi\bar{\rho}(0)G} \\ &= 1 - \frac{3c_1^2}{2\omega^2} \end{aligned} \tag{29}$$

Note that α is independent of the radius a. The constant c_1 may now be expressed in terms of α.

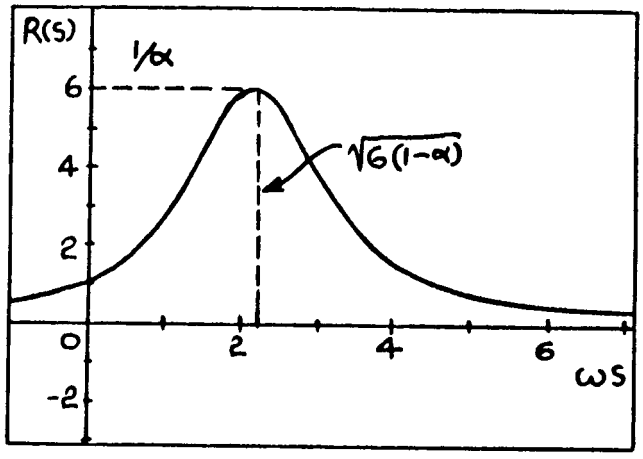
$$C_1 = \omega \sqrt{2/3\ (1-\alpha)} \tag{30}$$

Combining equation (30) with equation (24) we find

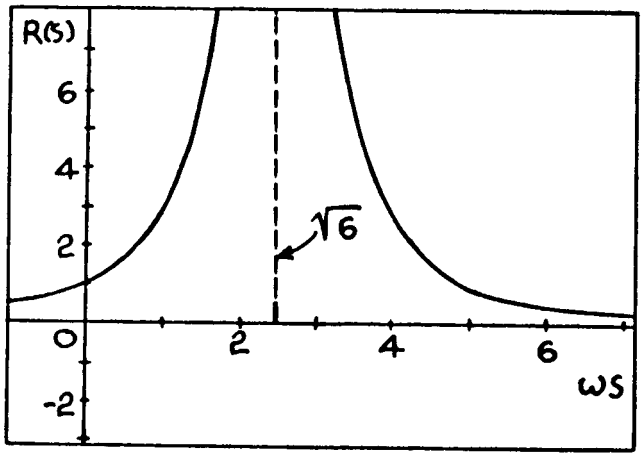
$$R(s) = \frac{1}{\frac{1}{6}(\omega s)^2 - \sqrt{\frac{2}{3}(1-\alpha)}\ \omega s + 1} \tag{31}$$

We may distinguish three cases depending upon whether α is positive, negative, or zero. These cases are presented graphically on the following page. Case i corresponds to the Einstein-Friedmann cosmological solutions which are spatially finite and positively curved. Case ii corresponds to the solution which is spatially infinite and flat. Case iii corresponds to the solutions which are spatially infinite and negatively curved. In the second two cases only those portions of the curves that can be reached continuously from s=0 have physical significance.

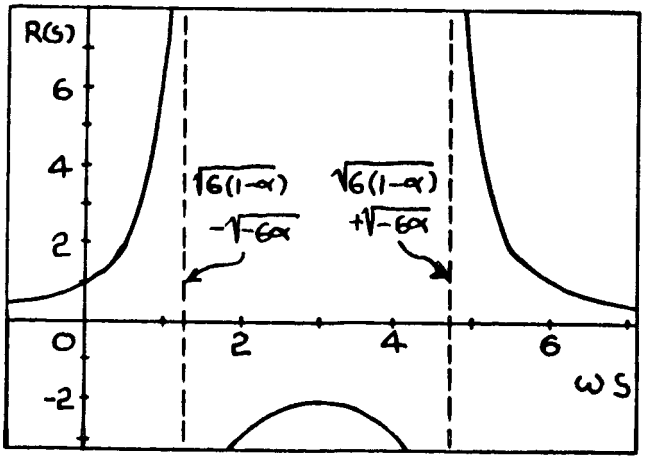
Observational evidence does not yet permit a choice to be made between the three possible cases. The flat universe appears most favorable to the development of irregularities along the lines suggested by the hypothesis of gravitational clustering. Consider a small spherical region in which the density is slightly greater than the average value. If this region is sufficiently isolated from the other irregularities so as to evolve independently it will



$\alpha > 0$ (1)



$\alpha = 0$ (11)



$\alpha < 0$ (111)

FIGURE 1

THE EXPANSION PARAMETER R AS A FUNCTION OF ωs

attain a minimum density and then fall back on itself. On the other hand the average density will approach zero asymptotically. Therefore the density contrast due to even slight initial irregularities will become very large.

It is also aesthetically satisfying to believe that the cosmological solution describing the actual universe is particularly simple. In a fully relativistic treatment the first and third solutions require a specification of an average radius of curvature, something not required of the second solution. This requirement of an additional parameter is usually obscured by writing the solutions in terms of dimensionless coordinates in which the radius of curvature is taken as the characteristic length.

In any event none of the following work depends upon which cosmological solution actually obtains.

CHAPTER III

THE STATISTICAL DESCRIPTION OF CLUSTERING

It is clear of course that because of the large number of particles involved a statistical treatment of the present problem is required. In such a treatment the universe is to be regarded as a particular realization of an ensemble of possible universes. Of course since the actual universe is quite unique the only averages having observational significance are space averages. Nevertheless, the introduction of the ensemble concept is natural for the following reasons.

Suppose one is given an exact description of the state of the universe at some initial time. One tends to divide this information into a part describing the gross structure, comprising such data as the density averaged over sufficiently large regions, and a part describing the details such as the precise arrangement of stars in individual galaxies. One intuitively expects the macroscopic information to be self-determinate. That is, if one is given two universes which are macroscopically equivalent but different in the details of their construction, this macroscopic equivalence should persist in the course of their evolution.

As a practical matter one cannot hope to obtain a

detailed description of the present state of the universe. Consequently one cannot do better than use the information available to characterize an ensemble of possible universes consistent with this information.

Finally, if the correlations between the irregularities of the cosmic distribution diminish sufficiently rapidly with increasing distance remote regions of space are statistically independent. Since the probability distribution is assumed statistically homogeneous such regions may be regarded as independent realizations of an ensemble. For this reason averaging over all space may be regarded as equivalent to averaging over the appropriate ensemble. The Birkhoff-Khintchine ergodic theorem (5) provides formal verification of this assumption.

For the foregoing reasons one is led to consider the evolution of ensembles of macroscopically similar universes. One must, however, define the concept of macroscopic equivalence somewhat more precisely.

In the study of plasma waves for example one adopts the viewpoint that variations on the scale of the wavelength are non-random. Thus, the average of the density over distances which are small compared to a wavelength but large

compared to the mean interparticle distance is considered to be a macroscopic feature of the distribution. An ensemble incorporating this viewpoint is constructed of representatives which are all similar on the scale of the wavelength.

On the other hand in studying homogeneous turbulence one views the turbulent eddies themselves as stochastic entities. In this case only quantities averaged over all space are considered to be macroscopic. An ensemble incorporating this viewpoint is constructed of representatives which may be quite different on the scale of the individual eddies but which give substantially equal results upon averaging over all space.

The first viewpoint may be called the fine view. It is appropriate to situations in which recognizable regularities exist on some scale of length which is large compared to the mean interparticle distance. The second viewpoint may be called the coarse view. It is appropriate to situations of gross statistical homogeneity. Both views are handled formally in the same way. One merely chooses different ensembles in the two cases.

On the whole the coarse view is more appropriate to the cosmological problem and will therefore be adopted.

I. THE DISTRIBUTION FUNCTIONS

Let us again consider the universe to be the limit of a distribution of N particles in a volume Ω as N and Ω tend toward infinity while $N/\Omega = C$ remains fixed. The state of such a system is given by specifying 6N variables, the N position vectors and the N velocity vectors. These may be considered to be the coordinates of a representative point in the phase space of the system. A statistical specification of the state of the system consists of assigning a probability to each subset of the phase space. We will assume for convenience that these probabilities are derivable from a density D, the Liouville probability density.

Recall now that in the original inertial system the position and velocity of the jth particle were denoted by $\vec{x}_j$ and $\vec{v}_j$. We had also introduced the co-moving coordinates $\vec{y}_j$ defined by

$$\vec{X}_j = Y_j R(s) \tag{32}$$

We may also define a velocity-like variable in the co-moving frame.

$$\vec{Z}_j = \frac{d\vec{Y}_j}{ds} \tag{33}$$

To see the physical significance of $\vec{z}_j$ note that the average velocity of expansion of the substratum is given by

$$\vec{\bar{V}} = \frac{1}{R}\frac{dR}{dt}\vec{x} = H\vec{x} \tag{34}$$

where H is Hubble's constant. Since

$$\vec{z}_j = \frac{d}{ds}\left(\frac{\vec{x}_j}{R}\right) = R^2\frac{1}{dt}\left(\frac{\vec{x}_j}{R}\right) = R\left(\vec{v}_j - \vec{x}_j H\right) \tag{35}$$

we see that $\vec{z}_j$ is the peculiar velocity scaled by R(s).

The state of the system of N particles may now be described by the quantities $(\vec{y}_1 \ldots \vec{y}_N, \vec{z}_1 \ldots \vec{z}_N)$. It will be convenient to use an abreviated notation. We will write (y,z) for $(\vec{y}_1 \ldots \vec{y}_N, \vec{z}_1 \ldots \vec{z}_N)$, $dydz$ for $d\vec{y}_1 \ldots d\vec{y}_N d\vec{z}_1 \ldots d\vec{z}_N$ etc. The probability that the representative point of the system is to be found in the region $dydz$ about (y,z) is then given by $D(y,z)dydz$. Since this probability is normalized we require

$$\int D(Y,z)\,dY\,dz = 1 \tag{36}$$

D satisfies the Liouville equation. Letting $\vec{F}_j$ be the total force on the jth particle this equation is

$$\frac{\partial D}{\partial s} + \sum_{j=1}^{N} \vec{z}_j \cdot \frac{\partial D}{\partial \vec{r}_j} + \sum_{j=1}^{N} \frac{\vec{F}_j}{m_j} \cdot \frac{\partial D}{\partial \vec{z}_j} = 0 \qquad (37)$$

At this point we will limit the discussion to systems of particles of identical mass. This is not a serious limitation because we are primarily interested in clustering on a scale of length much larger than the mean interparticle distance. In this limit, as we shall later see, the dynamics of the system are independent of the particulate nature of the distribution. We might mention however that one can treat the particles as equivalent and still allow for a variation in the individual particle masses by treating the mass of each particle as a random variable with a probability distribution independent of the particle index. In that way the system's state would be described by 7N variables, the additional N variables being the particle masses. D would then be a probability density in the 7N dimensional extended phase space.

Now the acceleration of the jth particle in the comoving frame is given by equation (22). Let the constant particle mass be M and note that $\bar{\rho}(0)$ may be written

$$\bar{\rho}(v) = MC \tag{38}$$

where C is the time independent number density of particles in the co-moving frame. Liouville's equation is thus rewritten as

$$\frac{\partial D}{\partial s} + \sum_j \vec{z}_j \cdot \frac{\partial D}{\partial \vec{r}_j} + \sum_{j,k}{}' \vec{F}^{(e)}_{j,k} \cdot \frac{\partial D}{\partial \vec{z}_j} + \sum_j \vec{F}^{(i)}_j \cdot \frac{\partial D}{\partial \vec{z}_j} = 0 \tag{39}$$

where the two-particle force exerted by particle k on particle j has been denoted by

$$F^{(e)}_{j,k} = -RMG \frac{(\vec{r}_j - \vec{r}_k)}{|\vec{r}_j - \vec{r}_k|^3} \tag{40}$$

and the single-particle background force by

$$F^{(i)}_j = \frac{4\pi}{3} RMGC \vec{r}_j \tag{41}$$

The prime over the second summation indicates that the terms j=k are to be omitted.

The probability density D obeys a single linear equation but one impossible to solve in any generality because of the large number of independent variables. It is clear however that we really do not want all of the information contained in D. The quantities which we may consider as macroscopic are all averages of sums of terms depending on only a few particles at a time. Consider two such quantities which play a role in the present problem.

In his continuum treatment Layzer introduced a quantity T_m which is one half the mass averaged square of the peculiar velocity. T_m is called the peculiar kinetic energy per unit mass and is defined by

$$T_m = \frac{\int_\Omega \rho(\vec{x})\, \tilde{V}^2(\vec{x})\, d\vec{x}}{2\int_\Omega \rho(\vec{x})\, d\vec{x}} \tag{42}$$

In the present particulate description we may define T_m as

$$T_m = \frac{\sum_{j=1}^{N} M \langle \tilde{V}_j^2 \rangle}{2\sum_{j=1}^{N} M} = \frac{1}{2NR^2} \sum_{j=1}^{N} \langle \tilde{z}_j^2 \rangle \tag{43}$$

This is a sum of terms depending on one particle at a time.

The total specific binding energy has been defined as

$$\mathcal{U}_m = \frac{-G}{2\bar{\rho}\Omega} \int_\Omega \int_\Omega \frac{\rho(\vec{x})\,\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d\vec{x}\, d\vec{x}' \tag{44}$$

In the particulate description this becomes

$$\mathcal{U}_m = \frac{-G}{NM} \sum_{i<j} \left\langle \frac{M^2}{|\vec{x}_i - \vec{x}_j|} \right\rangle = \frac{-GM}{RN} \sum_{i<j} \left\langle \frac{1}{|\vec{y}_i - \vec{y}_j|} \right\rangle \tag{45}$$

This is a sum of quantities depending on two particles at a time.

In general let A be a sum of terms depending on n

particles at a time.

$$A = \sum_{\rho} a(\vec{\gamma}_{j_1} \ldots \vec{\gamma}_{j_n}, \vec{z}_{j_1} \ldots \vec{z}_{j_n}) \tag{46}$$

where the summation extends over all ways n indices may be chosen from N without regard to order and a is a function which is symmetric in all particle indices. The average value of A is then given by

$$\begin{aligned} \langle A \rangle &= \int D(\gamma, z) \sum_{\rho} a(\vec{\gamma}_{j_1} \ldots \vec{\gamma}_{j_n}, \vec{z}_{j_1} \ldots \vec{z}_{j_n}) \, d\gamma \, dz \\ &= \frac{N!}{n!(N-n)!} \int D(\gamma, z) a(\gamma_1 \ldots \gamma_n, z_1 \ldots z_n) \, d\gamma \, dz \\ &= \frac{N!}{n!(N-n)!} \int \left[\int D(\gamma, z) \prod_{j=n+1}^{N} d\vec{\gamma}_j \, d\vec{z}_j \right] a(\vec{\gamma}_1 \ldots \vec{\gamma}_n, \vec{z}_1 \ldots \vec{z}_n) \prod_{m=1}^{n} d\vec{\gamma}_m \, d\vec{z}_m \end{aligned} \tag{47}$$

Here we have assumed, as will always be done, that the probability density is symmetric in all particle indices.

Let us now adopt the abreviated notation $a(\vec{y}_1 \ldots \vec{y}_N, \vec{z}_1 \ldots \vec{z}_N) \rightarrow a(1..N)$, $d\vec{y}_j d\vec{z}_j \rightarrow d(j)$ etc. Whenever a particle index appears as the argument of a function it is to be understood as representing both the corresponding velocity and coordinates. We now define the $f^{(n)}$, the n-particle distribution functions.

$$f^{(n)}(1 \ldots n) = \Omega^n \int D(1 \ldots N) \prod_{j=n+1}^{N} d(j) \tag{48}$$

Now for fixed n

$$\lim_{N\to\infty} \frac{N!}{(N-n)!} = N^n \tag{49}$$

Thus, for large N we may write

$$\langle A \rangle = \frac{C^n}{n!} \int \mathcal{S}^{(n)}(1\ldots n) a(1\ldots n)\, d(1)\ldots d(n) \tag{50}$$

Let us now note some important properties of the distribution functions. Since D is symmetric in the particle indices 1 through N $f^{(n)}$ is symmetric in the indices 1 through n. As n increases $f^{(n)}$ represents an increasingly fine description of the statistical state of the system. For $n' > n$ all the information contained in $f^{(n)}$ is contained in $f^{(n')}$ also. The distribution functions are not independent but are connected through the following relation.

$$\mathcal{S}^{(n)}(1\ldots n) = \Omega^{n-n'} \int \mathcal{S}^{n'}(1\ldots n')\, d(n+1)\ldots d(n') \tag{51}$$
$$n' > n$$

By virtue of the cosmological principle D(1...N) is a homogeneous function of position. That is, in the limit of infinite volume

$$D(\vec{r}_1+\vec{a}, \vec{r}_2+\vec{a}, \ldots \vec{r}_N+\vec{a}; \vec{z}_1 \ldots \vec{z}_N) = D(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N; \vec{z}_1 \ldots \vec{z}_N) \tag{52}$$

for arbitrary displacements **a**. The distribution functions are similarly homogeneous.

Consider a group of n particles distributed in two individually compact subgroups. Let particles 1...n' form one subgroup and particles n'+1...n the other. If the separation of the subgroups is large we may reasonably expect the subgroups to be statistically independent. That is, we expect the distribution function for the n particles to factor.

$$f^{(n)}(1\ldots n) \to f^{(n')}(1\ldots n')\, f^{(n-n')}(n'+1\ldots n) \tag{53}$$

We will in fact assume the rate of factorization to be sufficiently rapid so that the following integral relation holds.

$$\lim_{\Omega\to\infty} \frac{1}{\Omega}\int f^{(n)}(\vec{r}_1+\vec{x}, \vec{r}_2+\vec{x}, \ldots, \vec{r}_{n'}+\vec{x}, \vec{r}_{n'+1}, \vec{r}_{n'+2}, \ldots \vec{r}_n; \vec{z}_1 \ldots \vec{z}_n)\, d\vec{x} = f^{(n')}(1\ldots n') f^{(n-n')}(n'+1,\ldots,n) \tag{54}$$

To derive a set of equations for the distribution functions we multiply equation (39) by Ω^n and integrate with respect to $\vec{y}_{n+1}\ldots\vec{y}_N, \vec{z}_{n+1}\ldots\vec{z}_N$. the first term is simply the derivative of $f^{(n)}$ with respect to s. Since the distribution is bounded in space integration over $\vec{y}_{n+1}\ldots\vec{y}_N$ in the second term eliminates contributions corresponding to j=n+1 through j=N. Similarly, in the last term we assume that the probability distribution is such that D vanishes strongly with increasing peculiar velocity. Then integration over $\vec{z}_{n+1}\ldots\vec{z}_N$ will eliminate terms corresponding to j=n+1

through j=N. Again, in the third term we have contributions only from j=1 through j=n. We may break up the summation over k into a part corresponding to $k \leq n$ and a part corresponding to $k > n$. This last part may be written as

$$\Omega^N \sum_{j=1}^{n} \sum_{k=n+1}^{N} \int \vec{F}_{j,k}^{(2)} \cdot \frac{\partial D}{\partial \vec{z}_j} \prod_{\ell=n+1}^{N} d(\ell) = \frac{N-n}{\Omega} \sum_{j=1}^{n} \frac{\partial}{\partial \vec{z}_j} \cdot \int \vec{F}_{j,n+1}^{(1)} S^{(n+1)}(1..n+1) d(n+1) \qquad (55)$$

In the limit as N goes to infinity $(N-n)/\Omega$ goes to C, the particle concentration. In this limit the equations for the distribution functions are therefore given by

$$\frac{\partial}{\partial s} S^{(n)}(1..n) + \sum_{j=1}^{n} \vec{z}_j \cdot \frac{\partial}{\partial \vec{r}_j} S^{(n)}(1..n) + \sum_{j=1}^{n} \vec{F}_j^{(1)} \cdot \frac{\partial}{\partial \vec{z}_j} S^{(n)}(1..n)$$
$$+ \sum_{j,k=1}^{n}{}' \vec{F}_{j,k}^{(2)} \cdot \frac{\partial}{\partial \vec{z}_j} S^{(n)}(1..n) + C \sum_{j=1}^{n} \frac{\partial}{\partial \vec{z}_j} \cdot \int \vec{F}_{j,n+1}^{(1)} S^{(n+1)}(1..n+1) d(n+1) = 0 \qquad (56)$$

This set of equations is essentially that known as the BBGKY hierarchy after its originators Bogolioubov, Born, Green, Kirkwood, and Yvon.

II. THE GENERATING FUNCTIONAL

A number of statistical problems of interest are handled most easily in terms of a generating functional. Let us define the generating functional $\mathcal{L}[u]$ as follows.

$$\mathcal{L}[u] \equiv \left\langle \prod_{\ell=1}^{N} \left[1 + \tfrac{1}{2} u(\ell)\right] \right\rangle$$
$$= \int D(Y, z) \prod_{\ell=1}^{N} \left[1 + \tfrac{1}{2} u(\vec{r}_\ell, \vec{z}_\ell)\right] d\vec{r}\, d\vec{z} \qquad (57)$$

$\mathcal{L}$ is a functional of $u(\vec{y},\vec{z})$, an arbitrary function of a single position argument and a single velocity argument. It was first introduced into statistical mechanics by N. N. Bogolioubov (6). Notice that the functional is an ensemble average of a certain function of the N positions and N velocities. Let us expand this function in powers of u.

$$\prod_{\ell=1}^{N}\left[1+\frac{1}{c}u(\ell)\right] = 1+\frac{1}{c}\sum_{\ell_1}u(\ell_1)+\frac{1}{c^2}\sum_{\ell_1<\ell_2}u(\ell_1)u(\ell_2)+\frac{1}{c^3}\sum_{\ell_1<\ell_2<\ell_3}u(\ell_1)u(\ell_2)u(\ell_3)+\dots \qquad (58)$$

Each sum in this series is precisely of the form of the right hand side of equation (46). We may thus immediately apply equation (50) to determine the limiting form of the generating functional as N tends toward infinity.

$$\lim_{N\to\infty}\mathcal{L}[u] = 1+\sum_{n=1}^{\infty}\frac{1}{n!}\int f^{(n)}(1\dots n)\,u(1)\dots u(n)\,d(1)\dots d(n) \qquad (59)$$

The distribution function $f^{(n)}$ is simply the coefficient of the monomial functional of the nth degree in u divided by n! . We may also express this in terms of the functional derivatives of $\mathcal{L}$, the properties of which are summarized in appendix A. One determines immediately from equation (59) that in the limit of large N the distribution functions are given by

$$f^{(n)}(1\dots n) = \left.\frac{\delta^n\mathcal{L}[u]}{\delta u(1)\dots\delta u(n)}\right]_{u=0} \qquad (60)$$

Now $\mathcal{L}$ depends on the time s through the distribution functions. We may derive an equation for this time dependence by differentiating equation (59) with respect to s and then substituting equation (56). Invoking the symmetry of the distribution functions with respect to interchanges of particle indices we obtain the following result.

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial s} &= -\sum_{n=1}^{\infty}\frac{1}{n!}\int\Big[\sum_{j=1}^{n}\vec{z}_j\cdot\frac{\partial \zeta^{(n)}}{\partial \vec{r}_j}+\sum_{j=1}^{n}\vec{F}_j^{(1)}\cdot\frac{\partial \zeta^{(n)}}{\partial \vec{z}_j}+\sum_{j,k}{}'\vec{F}_{j,k}^{(2)}\cdot\frac{\partial \zeta^{(n)}}{\partial \vec{z}_j}\\ &\quad+c\sum_{j=1}^{n}\frac{\partial}{\partial \vec{z}_j}\cdot\int\vec{F}_{j,n+1}^{(2)}\zeta^{(n+1)}(1..n+1)d(n+1)\Big]u(1)..u(n)d(1)..d(n)\\ &= -\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\int\Big[\vec{z}_1\cdot\frac{\partial \zeta^{(n)}}{\partial \vec{r}_1}+\vec{F}_1^{(1)}\cdot\frac{\partial \zeta^{(n)}}{\partial \vec{z}_1}\Big]u(1)..u(n)d(1)..d(n)\\ &\quad+\sum_{n=2}^{\infty}\frac{1}{(n-2)!}\int\Big[u(2)\vec{F}_{1,2}^{(2)}\cdot\frac{\partial \zeta^{(n)}}{\partial \vec{z}_1}+c\cdot\frac{\partial \zeta^{(n)}}{\partial \vec{z}_1}\Big]u(1)u(3)..u(n)d(1)..d(n)\end{aligned} \tag{60a}$$

This equation may be expressed in terms of the first two functional derivatives of $\mathcal{L}$.

$$\frac{\delta \mathcal{L}}{\delta u(1)} = \sum_{n=1}^{\infty}\frac{1}{(n-1)!}\int\zeta^{(n)}(1..n)u(2)..u(n)d(2)..d(n) \tag{60b}$$

and

$$\frac{\delta^2 \mathcal{L}}{\delta u(1)\delta u(2)} = \sum_{n=2}^{\infty}\frac{1}{(n-2)!}\int\zeta^{(n)}(1..n)u(3)..u(n)d(3)..d(n) \tag{60c}$$

Thus

$$\begin{aligned}&\frac{\partial \mathcal{L}}{\partial s}+\int u(1)\vec{z}_1\cdot\frac{\partial}{\partial \vec{r}_1}\frac{\delta \mathcal{L}}{\delta u(1)}d(1)+\int u(1)\vec{F}_1^{(1)}\cdot\frac{\partial}{\partial \vec{z}_1}\frac{\delta \mathcal{L}}{\delta u(1)}d(1)\\ &+\iint u(1)u(2)\vec{F}_{1,2}^{(2)}\cdot\frac{\partial}{\partial \vec{z}_1}\frac{\delta^2 \mathcal{L}}{\delta u(1)\delta u(2)}d(1)d(2)+c\iint u(1)\vec{F}_{1,2}^{(2)}\cdot\frac{\partial}{\partial \vec{z}_1}\frac{\delta^2 \mathcal{L}}{\delta u(1)\delta u(2)}d(1)d(2)=0\end{aligned} \tag{61}$$

Another interesting functional, one which has been employed in the theory of random fields, is the characteristic functional of Hopf (7). This functional was applied originally to the study of hydrodynamic turbulence.

Suppose $\mu(y,z)$ is the density in position-velocity space of a distribution of matter. The characteristic functional associated with μ is defined to be

$$\Phi[w] = \left\langle e^{i\int \mu(\vec{r},\vec{s})\, w(\vec{r},\vec{s})\, d\vec{r}\, d\vec{s}} \right\rangle \tag{61a}$$

The n-position, n-velocity correlation functions are then expressible in terms of the functional derivatives of Φ evaluated at w=0.

$$\langle \mu(1) \ldots \mu(n) \rangle = \left(\frac{1}{i}\right)^n \frac{\delta^n \Phi}{\delta w(1) \ldots \delta w(n)} \Bigg]_{w=0} \tag{62}$$

In particular one may express the physically interesting two-point density correlation function as

$$\langle \rho(\vec{r}_1)\, \rho(\vec{r}_2) \rangle = -\iint \frac{\delta^2 \Phi}{\delta w(1)\, \delta w(2)} \Bigg]_{w=0} d\vec{s}_1\, d\vec{s}_2 \tag{63}$$

Let us now establish the connection between the characteristic functional for a distribution of mass points and the corresponding generating functional. For a collection of mass points at positions $\vec{y}_1 \ldots \vec{y}_N$ with velocities $\vec{z}_1 \ldots \vec{z}_N$ the function μ is given by

$$\mu(\vec{r},\vec{z}) = \sum_{j=1}^{N} M\,\delta(\vec{r}-\vec{r}_j)\,\delta(\vec{z}-\vec{z}_j) \tag{64}$$

Therefore

$$\begin{aligned}\Phi[w] &= \left\langle e^{i\int \mu(\vec{r},\vec{z})\,w(\vec{r},\vec{z})\,d\vec{r}\,d\vec{z}} \right\rangle \\ &= \left\langle e^{i\sum_{j=1}^{N} M\,w(\vec{r}_j,\vec{z}_j)} \right\rangle \\ &= \left\langle \prod_{j=1}^{N} e^{iMw(j)} \right\rangle \\ &= \left\langle \prod_{j=1}^{N} \left[1-\tfrac{1}{c}\{c(e^{iMw(j)}-1)\}\right] \right\rangle \\ &= \mathcal{L}\left[c(e^{iMw}-1)\right]\end{aligned} \tag{65}$$

We see that the characteristic functional is obtained from the generating functional by a simple change of variables.

III. THE CORRELATION FUNCTIONS

Another set of quantities which describe the clustering of particles are the correlation functions, sometimes known as the Ursell-Mayer functions. These are constructed from the distribution functions but seem in a number of ways to be more fundamental. One indication of this is that the correlation functions are independent quantities whereas the distribution functions are not. Also, as calculation of these two sets of functions for specific models will later indicate, the correlation functions are often the simpler in form. In particular, as the calculations based upon a

model of independent clusters will indicate, if the highest order correlations involve n' particles the n-particle correlation functions all vanish for $n > n'$.

Let us denote the n-particle correlation function by $g^{(n)}(\vec{y}_1 \ldots \vec{y}_n; \vec{z}_1 \ldots \vec{z}_n) = g^{(n)}(1 \ldots n)$. The correlation functions may be defined recursively in terms of the distribution functions as follows.

$$f^{(1)}(1) = g^{(1)}(1) \tag{66a}$$

$$f^{(2)}(1,2) = g^{(2)}(1,2) + g^{(1)}(1)\, g^{(1)}(2) \tag{66b}$$

$$\begin{aligned} f^{(3)}(1,2,3) = {} & g^{(3)}(1,2,3) + g^{(2)}(1,2) g^{(1)}(3) + g^{(2)}(1,3) g^{(1)}(2) \\ & + g^{(2)}(3,1) g^{(1)}(2) + g^{(1)}(1) g^{(1)}(2) g^{(1)}(3) \end{aligned} \tag{66c}$$

etc. In general each of the $f^{(n)}$ is expressed as a sum of terms, each term corresponding to a different partition of the set of indices $1 \ldots n$.

The expressions for the $f^{(n)}$ given in equations (66) are closely related to the cluster expansion of equilibrium statistical mechanics. The first applications of this expansion to kinetic theory appear to have been made by M. Green (8) and Brout (9). Green in particular strongly emphasized the independence of the correlation functions.

It is common practice to denote the single-particle velocity distribution function by the symbol ϕ . If the single-particle distribution function depends on position as well as velocity the velocity distribution function is defined to be

$$\phi(\vec{z}) = \lim_{\Omega \to \infty} \frac{1}{\Omega} \int f^{(1)}(\vec{v}, \vec{z})\, d\vec{v} \tag{67}$$

In the present case the cosmological principle dictates that $f^{(1)}$ is independent of position. Therefore

$$\phi(\vec{z}) = f^{(1)}(\vec{v}, \vec{z}) = \partial^{(1)}(\vec{v}, \vec{z}) \tag{68}$$

In order to facilitate the investigation of the properties of the correlation functions it is useful to first obtain their generating functional. Let $A[u]$ be the natural logarithm of $\mathcal{L}[u]$, the distribution generating functional. The first three functional derivatives of $\mathcal{L}[u]$ may therefore be expressed as

$$\mathcal{L} = e^{A} \tag{69}$$

$$\frac{\delta \mathcal{L}}{\delta u(1)} = e^{A} \frac{\delta A}{\delta u(1)} \tag{70}$$

$$\frac{\delta^2 \mathcal{L}}{\delta u(1)\, \delta u(2)} = e^{A} \left[\frac{\delta^2 A}{\delta u(1)\, \delta u(2)} + \frac{\delta A}{\delta u(1)} \frac{\delta A}{\delta u(2)} \right] \tag{71}$$

$$\frac{\delta^3 \mathcal{L}}{\delta u(1)\, \delta u(2)\, \delta u(3)} = e^{A} \left[\frac{\delta^3 A}{\delta u(1)\, \delta u(2)\, \delta u(3)} + \frac{\delta^2 A}{\delta u(1)\, \delta u(2)} \frac{\delta A}{\delta u(3)} + \frac{\delta^2 A}{\delta u(2)\, \delta u(3)} \frac{\delta A}{\delta u(1)} + \frac{\delta^2 A}{\delta u(3)\, \delta u(1)} \frac{\delta A}{\delta u(2)} + \frac{\delta A}{\delta u(1)} \frac{\delta A}{\delta u(2)} \frac{\delta A}{\delta u(3)} \right] \tag{72}$$

Note that

$$\mathcal{Z}[0] = \left\langle \prod_{\ell=1}^{N} [1 + \tfrac{1}{2} u(\ell)] \right\rangle \Big]_{u=0} = 1 \tag{73}$$

Evaluating equations (70), (71), and (72) at u=0, substituting for the derivatives of $\mathcal{Z}$ the corresponding distribution functions, and then solving for the derivatives of A with the aid of equations (66) we obtain

$$g^{(1)}(1) = \frac{\delta A}{\delta u(1)} \Big]_{u=0} \tag{74}$$

$$g^{(2)}(1,2) = \frac{\delta^2 A}{\delta u(1)\, \delta u(2)} \Big]_{u=0} \tag{75}$$

$$g^{(3)}(1,2,3) = \frac{\delta^3 A}{\delta u(1)\, \delta u(2)\, \delta u(3)} \Big]_{u=0} \tag{76}$$

It is not difficult to convince oneself that in general the following relation holds.

$$g^{(n)}(1,2,3) = \frac{\delta^n A}{\delta u(1) \ldots \delta u(n)} \Big]_{u=0} \tag{77}$$

A then is the generating functional for the correlation functions.

One of the reasons mentioned for considering the correlation functions to be more fundamental than the distribution functions is their independence. This may be seen as follows.

As we have seen the various distribution functions are not independent since they must satisfy the identity

$$f^{(n)}(1\ldots n) = \frac{1}{\Omega}\int f^{(n+1)}(1\ldots n+1)\, d(n+1) \tag{78}$$

Let us express this identity in terms of the correlation functions. The most convenient way of doing this is to proceed by way of the generating functional. Consider the following identity satisfied by the generating functional $\mathcal{L}[u]$.

$$\begin{aligned}
&\frac{1}{\Omega}\int \frac{\delta \mathcal{L}}{\delta u(1)}\, d(1) \\
&\quad = \frac{1}{\Omega}\int \frac{\delta}{\delta u(1')}\left[1 + \sum_{n=1}^{\infty}\frac{1}{n!}\int f^{(n)}(1\ldots n)\, u(1)\ldots u(n)\, d(1)\ldots d(n)\right] d(1') \\
&\quad = \frac{1}{\Omega}\int\left[f^{(1)}(1) + \sum_{n=2}^{\infty}\frac{1}{(n-1)!}\int f^{(n)}(1\ldots n)\, u(2)\ldots u(n)\, d(2)\ldots d(n)\right] d(1) \qquad (79)\\
&\quad = 1 + \sum_{n=2}^{\infty}\frac{1}{(n-1)!}\int f^{(n-1)}(2\ldots n)\, u(2)\ldots u(n)\, d(2)\ldots d(n) \\
&\quad = \mathcal{L}
\end{aligned}$$

This identity is precisely equivalent to the set of equations (78). Expressed in terms of A the identity becomes

$$\begin{aligned}
&\frac{1}{\Omega}\int e^{A}\frac{\delta A}{\delta u(1)}\, d(1) = e^{A} \\
&\frac{1}{\Omega}\int \frac{\delta A}{\delta u(1)}\, d(1) = 1
\end{aligned} \tag{80}$$

Evaluating equation (80) at u=0 we find

$$\frac{1}{\Omega}\int g^{(1)}(1)\, d(1) = 1 \tag{81}$$

which is not surprising since $g^{(1)}=f^{(1)}$, a normalized probability density. Differentiating equation (80) n-1 times and then setting u=0 we find

$$\frac{1}{\Omega}\int g^{(n)}(1\ldots n)\,d\omega = 0 \qquad (82)$$

Equation (82) represents equation (78) expressed in terms of the correlation functions. We see that the identity now affects the various $g^{(n)}$ separately; it does not couple them. The correlation functions are, in this sense, independent. Their independence is not quite complete however. Since the distribution functions are essentially probability densities they are necessarily non-negative. Thus, each of the right hand sides of equations (66) are constrained to be non-negative. This set of inequalities obeyed by the $g^{(n)}$ limits their complete independence.

Since the distribution functions depend upon the time through the dynamics of the system so too do the correlation functions. We shall now derive the equations governing this time dependence. Let us therefore substitute in equation (61) the expressions (70) and (71) to obtain first the equation for $A[u]$.

$$\frac{\partial A}{\partial s} + \int u(1)\vec{z}_1 \cdot \frac{\partial}{\partial \vec{y}_1} \frac{\delta A}{\delta u(1)} d(1) + \int u(1)\vec{F}^{(1)}_{1,1} \cdot \frac{\partial}{\partial \vec{z}_1} \frac{\delta A}{\delta u(1)} d(1)$$
$$+ \iint u(1)u(2)\vec{F}^{(1)}_{1,2} \cdot \frac{\partial}{\partial \vec{z}_1} \left[\frac{\delta^2 A}{\delta u(1)\delta u(2)} + \frac{\delta A}{\delta u(1)} \frac{\delta A}{\delta u(2)} \right] \qquad (83)$$
$$+ c \iint u(1)\vec{F}^{(1)}_{1,2} \cdot \frac{\partial}{\partial \vec{z}_1} \left[\frac{\delta^2 A}{\delta u(1)\delta u(2)} + \frac{\delta A}{\delta u(1)} \frac{\delta A}{\delta u(2)} \right] = 0$$

To obtain the equations for the correlation functions we merely differentiate this equation n times with respect to u(1)...u(n) and then set u=0. In carrying out this differentiation it is best to replace the dummy variables $\vec{y}_1$, $\vec{y}_2$, $\vec{z}_1$, and $\vec{z}_2$ appearing in equation (83) by $\vec{y}_{n+1}$, $\vec{y}_{n+2}$, $\vec{z}_{n+1}$, and $\vec{z}_{n+2}$ so that no confusion arises with the arguments of $g^{(n)}(1\ldots n)$. Then, using the properties of the functional derivative given in the first appendix, the n-fold derivative of equation (83) is easily calculated. The equations for the correlation functions are found to be

$$\frac{\partial g^{(n)}}{\partial s} + \sum_{j=1}^{n} \vec{z}_j \cdot \frac{\partial}{\partial \vec{y}_j} g^{(n)} + \sum_{j=1}^{n} \vec{F}^{(1)}_{j,1} \cdot \frac{\partial}{\partial \vec{z}_j} g^{(n)}$$
$$+ \sum_{j,k=1}^{n}{}' \vec{F}^{(1)}_{j,k} \cdot \frac{\partial}{\partial \vec{z}_j} \left[g^{(n)}(1\ldots n) + \sum_p g^{(n)}(\ldots j\ldots) g^{(n-n)}(\ldots k\ldots) \right] \qquad (84)$$
$$+ c \sum_{j=1}^{n} \int \vec{F}^{(1)}_{j,n+1} \cdot \frac{\partial}{\partial \vec{z}_j} \left[g^{(n+1)}(1\ldots n+1) + \sum_p g^{(n)}(\ldots j\ldots) g^{(n+1-n)}(\ldots n+1) \right] d(n+1) = 0$$

where the quantity $\sum_p g^{(n)}(\ldots j\ldots) g^{(n-n)}(\ldots k\ldots)$ represents a sum of products of pairs of g's, each product corresponding to a partition of 1...n into two subsets, one containing the

index j and the other containing the index k. For example, if n=4, j=1, and k=2, the summation is explicitly given by

$$\sum_{P} g^{(n)}(1..)g^{(4-n)}(..2..) = g^{(3)}(1,3,4)g^{(1)}(2) + g^{(3)}(2,3,4)g^{(1)}(1) + g^{(2)}(1,3)g^{(2)}(2,4) + g^{(2)}(1,4)g^{(2)}(2,3) \tag{85}$$

Similarly, the other such summation in equation (84) corresponds to partitions of the set 1...n+1 into two parts, one part containing the index j, the other containing the index n+1.

Now, the two-particle force $\vec{F}^{(2)}$ and the single-particle background force $\vec{F}^{(1)}$ are connected through the requirement that the background force just compensate for the average sum of two-body forces. Since the average density of particles is C this relationship may be expressed as

$$C\int \vec{F}^{(2)}_{j,2}\, d\vec{Y}_2 + \vec{F}^{(1)}_j = 0 \tag{86}$$

One easily verifies that for the forces given by equations (40) and (41) this relation holds providing the correct limiting proceedure is used.

We may now use equation (86) together with the requirement that the single-particle correlation function be independent of position to simplify equations (84). Consider the last term of these equations. In the summation over partitions of 1...n+1 there is a contribution of the

form $g^{(n)}(1..n)g^{(1)}(n+1)$. Evaluating the contribution to the last term of equation (84) from this product we find

$$
\begin{aligned}
& C\sum_j \int \vec{F}^{(2)}_{j,n+1}\cdot\frac{\partial}{\partial \vec{z}_j}\,\vartheta^{(n)}(1\ldots n)\,\vartheta^{(1)}(n+1)\,d(n+1) \\
&\quad = C\sum_j \left[\int F^{(2)}_{j,n+1}\,d\vec{Y}_{n+1}\right]\cdot\frac{\partial}{\partial \vec{z}_j}\,\vartheta^{(n)}(1..n)\left[\int \vartheta^{(1)}(n+1)\,d\vec{z}_{n+1}\right] \qquad (87)\\
&\quad = -\sum_j \vec{F}^{(1)}_j\cdot\frac{\partial}{\partial \vec{z}_j}\,\vartheta^{(n)}(1\ldots n)
\end{aligned}
$$

This term just cancels the third term of equation (84). We may thus rewrite that equation as

$$
\begin{aligned}
& \frac{\partial \vartheta^{(n)}}{\partial s} + \sum_{j=1}^{n}\vec{z}_j\cdot\frac{\partial \vartheta^{(n)}}{\partial Y_j} \\
&\quad + \sum_{j,k=1}^{n}{}'\vec{F}^{(2)}_{j,k}\cdot\frac{\partial}{\partial \vec{z}_j}\left[\vartheta^{(n)}(1..n)+\sum_p \vartheta^{(n)}(..j..)\vartheta^{(n-n)}(..k..)\right] \qquad (88)\\
&\quad + C\sum_{j=1}^{n}\int \vec{F}^{(2)}_{j,n+1}\cdot\frac{\partial}{\partial \vec{z}_j}\left[\vartheta^{(n+1)}(1..n+1)+\sum_p{}'\vartheta^{(n)}(..j..)\vartheta^{(n+1-n)}(..n+1)\right]d(n+1)=0
\end{aligned}
$$

where the prime over the second partition sum indicates that the term $g^{(n)}(1..n)g^{(1)}(n+1)$ is to be omitted.

Let us note that since the force $\vec{F}^{(2)}$ is linear in the particle mass M, in the continuum limit ($M\to 0$, $C\to\infty$, MC=constant) the third term of equation (88) vanishes while the fourth term survives. The third term of the equation is therefore the part that describes relaxation due to close encounters.

Before leaving this chapter let us turn to the

expression of T_m, the specific peculiar kinetic energy, U_m, the specific peculiar binding energy, and $\mathcal{S}(k)$, the clustering spectrum, in terms of the correlation functions.

The specific peculiar kinetic energy was given in equation (44) as

$$T_m = \frac{R(s)^{-2}}{2N} \sum_{j=1}^{N} \langle z_j^2 \rangle \tag{89}$$

This average is computed according to the general prescription of equation (50).

$$T_m = \frac{R(s)^{-2}}{2N} C \int z_1^2 \, \zeta^{(1)}(\vec{r}_1, \vec{z}_1) \, d\vec{r}_1 \, d\vec{z}_1 \tag{90}$$

Using the notation of equation (61) we obtain

$$\begin{aligned} T_m &= \frac{R(s)^{-2}}{2N} C\Omega \int \phi(\vec{z}_1) z_1^2 \, d\vec{z}_1 \\ &= \frac{R(s)^{-2}}{2} \int \phi(\vec{z}_1) z_1^2 \, d\vec{z}_1 \end{aligned} \tag{91}$$

Consider now the total specific binding energy. This is given by equation (45) as

$$\begin{aligned} U_m &= \frac{-GMR(s)^{-1}}{N} \sum_{i<j} \left\langle \frac{1}{|\vec{r}_i - \vec{r}_j|} \right\rangle \\ &= -\frac{C^2}{2} \frac{GMR^{-1}}{N} \int \zeta^{(2)}(\vec{r}_1 - \vec{r}_2; \vec{z}_1, \vec{z}_2) \frac{1}{|\vec{r}_1 - \vec{r}_2|} d\vec{r}_1 \, d\vec{r}_2 \, d\vec{z}_1 \, d\vec{z}_2 \\ &= -\frac{C}{2} GMR^{-1} \int \zeta^{(2)}(\vec{r}; \vec{z}_1, \vec{z}_2) \frac{1}{r} \, d\vec{r} \, d\vec{z}_1 \, d\vec{z}_2 \end{aligned} \tag{92}$$

$$= -\frac{c}{2} G M R^{-1} \int [g^{(2)}(\vec{r}; \vec{z}_1, \vec{z}_2) + g^{(1)}(\vec{z}_1) g^{(1)}(\vec{z}_2)] \frac{1}{r} d\vec{r}\, d\vec{z}_1\, d\vec{z}_2$$

$$= -\frac{c}{2} G M R^{-1} \left[\int g^{(2)}(\vec{r}; \vec{z}_1, \vec{z}_2) \frac{1}{r} d\vec{r}\, d\vec{z}_1\, d\vec{z}_2 + \int \frac{1}{r} d\vec{r} \right]$$

In the limit of infinite volume the last term of equation (92) diverges. Nevertheless, since it is independent of the clustering we shall regard it as the zero point of the potential and consequently ignore it. The remainder will be called the specific peculiar binding energy U_m.

$$U_m = -\frac{c}{2} G M R(s)^{-1} \int g^{(2)}(\vec{r}; \vec{z}_1, \vec{z}_2) \frac{1}{r} d\vec{r}\, d\vec{z}_1\, d\vec{z}_2 \qquad (93)$$

Let us note that equation (93) does not quite coincide with Layzer's definition given by equation (6). In the first place his definition is inapplicable to a particulate distribution since for such a distribution the mean square of the density fluctuations is infinite. Another difference is that Layzer's peculiar binding energy is always non-positive. This may be easily verified in terms of the Fourier transform of the peculiar part of the density. This transform exists for finite Ω although it is not defined in the limit of infinite volume.

$$
\begin{aligned}
U_m' &= \lim_{\Omega\to\infty} \frac{-G}{2\bar{\rho}\Omega} \int \frac{\tilde{\rho}(\vec{x})\,\tilde{\rho}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d\vec{x}\, d\vec{x}' \\
&= \lim_{\Omega\to\infty} \frac{-G}{2\bar{\rho}\Omega} \int e^{2\pi i[\vec{\xi}\cdot\vec{x} + \vec{\xi}'\cdot\vec{x}' + \vec{\xi}''\cdot(\vec{x}-\vec{x}')]} \frac{\bar{\tilde{\rho}}(\vec{\xi})\,\bar{\tilde{\rho}}(\vec{\xi}')}{\pi \xi''^2}\, d\vec{\xi}\, d\vec{x}\, d\vec{x}'\, d\vec{\xi}'\, d\vec{\xi}'' \\
&= \lim_{\Omega\to\infty} \frac{-G}{2\bar{\rho}\Omega} \int \delta(\vec{\xi}+\vec{\xi}'')\,\delta(\vec{\xi}'-\vec{\xi}'')\, \frac{\bar{\tilde{\rho}}(\vec{\xi})\,\bar{\tilde{\rho}}(\vec{\xi}')}{\pi \xi''^2}\, d\vec{\xi}\, d\vec{\xi}'\, d\vec{\xi}'' \qquad (94) \\
&= \lim_{\Omega\to\infty} \frac{-G}{2\bar{\rho}\Omega} \int \frac{\bar{\tilde{\rho}}(\vec{\xi})\,\bar{\tilde{\rho}}(-\vec{\xi})}{\pi \xi^2}\, d\vec{\xi} \\
&= \lim_{\Omega\to\infty} \frac{-G}{2\bar{\rho}\Omega} \int \frac{|\bar{\tilde{\rho}}(\vec{\xi})|^2}{\pi \xi^2}\, d\vec{\xi} \\
&\leq 0
\end{aligned}
$$

U_m, the peculiar specific binding energy defined by equation (93) vanishes for a Poisson distribution (or indeed for any distribution in which two-particle correlations are absent) but may otherwise take on either sign. In any event, although the peculiar binding energy defined here differs from Layzer's it will be shown in chapter V that U_m and T_m satisfy the same energy equation first derived by Irvine (4) for Layzer's quantities.

We wish also to define a clustering spectrum appropriate to particulate distributions. In order to obtain a quantity closely corresponding to Layzer's spectrum as given by equation (8) let us express the peculiar binding energy in terms of the inertial coordinate $\vec{x}$ instead of the co-

moving coordinate y.

$$U_m = -\frac{MCG}{2R(3)}\int\left(\int g^{(2)}\left(\frac{\vec{x}}{R};\vec{z},\vec{z}_2\right)d\vec{z}_1 d\vec{z}_2\right)\frac{1}{x}\,d\vec{x} \tag{95}$$

The clustering spectrum will be taken to be proportional to the Fourier transform of the quantity in parenthesis in the equation above.

$$\mathcal{S}(\mathcal{R}) = \text{const.}\times\int e^{-2\pi i\vec{\mathcal{R}}\cdot\vec{x}}\,g^{(2)}\left(\frac{\vec{x}}{R};\vec{z},\vec{z}_2\right)d\vec{z}_1 d\vec{z}_2 d\vec{x} \tag{96}$$

Let

$$\bar{g}^{(2)}(\vec{\mathcal{R}};\vec{z},\vec{z}_2) = \int e^{-2\pi i\vec{\mathcal{R}}\cdot\vec{x}}\,g^{(2)}(\vec{Y};\vec{z},\vec{z}_2)\,d\vec{Y} \tag{97}$$

Then

$$\mathcal{S}(\mathcal{R}) = \text{const}\cdot R(3)^2\int\bar{g}^{(2)}(\vec{\mathcal{R}}R;\vec{z},\vec{z}_2)\,d\vec{z}_1 d\vec{z}_2 \tag{98}$$

The constant is evaluated by requiring that $\mathcal{S}(k)$ like $\mathcal{S}^L(k)$ integrate to unity.

$$\begin{aligned}\int\mathcal{S}^L(\mathcal{R})\,d\vec{\mathcal{R}} &= f(0)\\ &= \frac{\langle\tilde{\delta}^2\rangle}{\langle\tilde{\delta}^2\rangle}\\ &= 1\end{aligned} \tag{99}$$

Thus

$$\begin{aligned}\mathcal{S}(\mathcal{R}) &= \frac{\int\bar{g}^{(2)}(\vec{\mathcal{R}}R;\vec{z},\vec{z}_2)\,d\vec{z}_1 d\vec{z}_2}{\int\bar{g}^{(2)}(\vec{\mathcal{R}}R;\vec{z},\vec{z}_2)\,d\vec{z}_1 d\vec{z}_2 d\vec{\mathcal{R}}}\\ &= \frac{R^3\int\bar{g}^{(2)}(\vec{\mathcal{R}}R;\vec{z},\vec{z}_2)\,d\vec{z}_1 d\vec{z}_2}{\int\bar{g}^{(2)}(\vec{\mathcal{R}};\vec{z},\vec{z}_2)\,d\vec{z}_1 d\vec{z}_2 d\vec{\mathcal{R}}}\end{aligned} \tag{100}$$

$$= \frac{R^3 \int \bar{g}^{(2)}(\vec{k}R; z, z_2)\, d\vec{z}_1 d\vec{z}_2}{\int g^{(2)}(0; z, z_2)\, d\vec{z}_1 d\vec{z}_2}$$

Note that because of the assumption of statistical isotropy $\mathcal{S}(k)$ depends on the magnitude of $\vec{k}$ but not its direction.

The peculiar binding energy is expressed in terms of the clustering spectrum as

$$U_m = -\frac{GM}{2R^3} \int g^{(2)}(0; \vec{z}_1, \vec{z}_2)\, dz_1 dz_2 \int \mathcal{S}(k) \frac{1}{\pi k^2}\, d\vec{k}$$
$$= \frac{-2\bar{\rho}(0) G}{R^3} \int g^{(2)}(0; \vec{z}_1, \vec{z}_2)\, d\vec{z}_1 d\vec{z}_2 \int_0^\infty \mathcal{S}(k)\, dk \qquad (101)$$

In the co-moving coordinate system it is probably more convenient to deal with a somewhat differently normalized clustering spectrum.

$$S(k) = \int \bar{g}^{(2)}(\hat{k}; \vec{z}_1, \vec{z}_2)\, d\vec{z}_1 d\vec{z}_2 \qquad (102)$$

Thus

$$\mathcal{S}(k) = \frac{R^3}{\int g^{(2)}(0; z, z_2)\, d\vec{z}_1 d\vec{z}_2} S(kR) \qquad (103)$$

In terms of this quantity U_m is given by

$$U_m = \frac{-2\bar{\rho}(0) G}{R(s)} \int_0^\infty S(k)\, dk \qquad (104)$$

CHAPTER IV

CLUSTERING MODELS

Before proceeding to the difficult problem of the actual dynamical evolution of the correlation functions it is useful to first gain a somewhat deeper understanding of their nature. For this reason we now turn to the study of particular statistical distributions.

I. THE GENERALIZED POISSON DISTRIBUTION

Consider first a distribution of particles constructed in the following manner. Let $p(\vec{y},\vec{z})$ be a definite function of position and velocity satisfying the normalization condition

$$\frac{1}{\Omega}\int p(\vec{y},\vec{z})\,d\vec{y}\,d\vec{z} = 1 \tag{105}$$

Let the probability density of finding any specified particle at the location $\vec{y}$ with the velocity $\vec{z}$ be $p(\vec{y},\vec{z})/\Omega$, independent of the index of the particle. If the scale of spatial variation of p is large compared to the mean interparticle distance almost all realizations will correspond to the usual concept of a spatially uncorrelated distribution with an average density $Np(\vec{y},\vec{z})/\Omega = Cp(\vec{y},\vec{z})$. For such distributions the generating functional is very easily calculated. In the limit of large N we have the following.

$$\begin{aligned} \mathcal{Z}[u] &= \int D(r,z) \prod_{j=1}^{N} [1 + \tfrac{1}{2} u(j)]\, d(1)\ldots d(N) \\ &= \int \frac{1}{\Omega^N} \prod_{j=1}^{N} P(j)[1 + \tfrac{1}{2} u(j)]\, d(1)\ldots d(N) \\ &= \left[\frac{1}{\Omega}\int P(j)[1 + \tfrac{1}{2} u(j)]\, d(j)\right]^N \\ &= \left[1 + \frac{1}{\Omega c}\int P(j)\, u(j)\, d(j)\right]^N \\ &\approx e^{\int P(1) u(1) d(1)} \end{aligned} \tag{106}$$

Now since

$$\begin{aligned} \mathcal{Z}[u] &= e^{A[u]} \\ &= e^{\sum_{n=1}^{\infty} \frac{1}{n!} \int g^{(n)}(1\ldots n)\, u(1)\ldots u(n)\, d(1)\ldots d(n)} \end{aligned} \tag{107}$$

we may immediately identify the correlation functions as the coefficients of the various powers of u in the exponent of the functional. Thus, in the present case

$$\begin{aligned} g^{(1)}(r_1, z_1) &= P(r_1, z_1) \\ g^{(n)}(1\ldots n) &= 0 \qquad\qquad n > 1 \end{aligned} \tag{108}$$

The distribution just described does not embody the cosmological principle because different points in space are treated differently. We may remedy this defect by

treating $p(\vec{y},\vec{z})$ itself as a statistically homogeneous and isotropic random field. The distribution of particles is now to be thought of as being constructed in two steps. First a function $p(\vec{y},\vec{z})$ is selected from an ensemble of such functions and then the particles are independently distributed with a probability density given by the particular function chosen.

The generating functional for this generalized Poisson distribution is obtained by averaging equation (107) over realizations of p.

$$\mathcal{L}[u] = \left\langle e^{\int p(\vec{r},\vec{s})\, u(\vec{r},\vec{z})\, d\vec{r}\, d\vec{z}} \right\rangle \tag{109}$$

But p, being a random field, has associated with it a characteristic functional $\Phi[w]$.

$$\Phi[w] = \left\langle e^{i\int p(\vec{r},\vec{z})\, w(\vec{r},\vec{z})\, d\vec{r}\, d\vec{z}} \right\rangle \tag{110}$$

Therefore the generating functional of the particle distribution and the characteristic functional of the random field p are related by

$$\mathcal{L}[u] = \Phi\left[\frac{1}{i}u\right] \tag{111}$$

It was mentioned in the last chapter that the peculiar binding energy given by equation (93) may, in general,

take on either sign. For the generalized Poisson distribution, however, U_m is never positive. To see this we split $p(\vec{y},\vec{z})$ into its average part $\bar{p}(z)$ and a part $\tilde{p}(y,z)$ the average of which vanishes. The one-particle and two-particle distribution functions are then

$$f^{(1)}(1) = \bar{P}(1) \tag{112}$$

and

$$f^{(2)}(1,2) = \bar{P}(1)\bar{P}(2) + \langle \tilde{P}(1)\tilde{P}(2) \rangle \tag{113}$$

The two particle correlation function is

$$g^{(2)}(1,2) = \langle \tilde{P}(1)\tilde{P}(2) \rangle \tag{114}$$

Finally, the quantity U_m may be written

$$U_m = -\frac{G}{2} M R^{-1} \int \frac{\langle \tilde{P}(\vec{Y}',\vec{z})\tilde{P}(\vec{Y}+\vec{Y}',\vec{z}') \rangle}{Y} d\vec{Y}\, d\vec{z}\, d\vec{z}' \tag{115}$$

If the correlations of p are of finite range the ensemble average of equation (115) may be replaced by a space average and the specific binding energy will then be precisely of the form of equation (94). We then conclude that in this case U_m is non-positive.

One interesting specialization of this generalized Poisson model results from the choice of p as a joint-

Gaussian random field (appendix B). This is not strictly permissible for with a Gaussian distribution there is a finite probability that p will assume a negative value at any given point. Since p is to be a probability density this is clearly inadmissible. Nevertheless, if the fluctuations of p from its mean are sufficiently small compared to $\bar{p}$ this objection cannot be of practical significance.

As is shown in appendix B the characteristic functional for a joint-Gaussian random field is

$$\Phi[w] = e^{i\int \bar{p}(1)w(1)d(1) - \frac{1}{2}\int \langle \tilde{p}(1)\tilde{p}(2)\rangle w(1)w(2)d(1)d(2)} \tag{116}$$

The generating functional for the distribution of particles is thus

$$\mathcal{L}[u] = e^{\int \bar{p}(1)u(1)d(1) + \frac{1}{2}\int \langle \tilde{p}(1)\tilde{p}(2)\rangle u(1)u(2)d(1)d(2)} \tag{117}$$

and we conclude that for this model the correlation functions are

$$g^{(1)}(\vec{r}_1, \vec{z}_1) = \bar{p}(\vec{r}_1, \vec{z}_1) \tag{118a}$$

$$g^{(2)}(\vec{r}_1\vec{r}_2; \vec{z}_1\vec{z}_2) = \langle \tilde{p}(\vec{r}_1, \vec{z}_1)\tilde{p}(\vec{r}_2, \vec{z}_2)\rangle \tag{118b}$$

$$g^{(n)}(1 \ldots n) = 0 \qquad n > 2 \tag{118c}$$

II. HIERARCHICAL CLUSTERING

We have already seen in chapter I how hierarchical distributions play a special role in the hypothesis of gravitational clustering. For this reason we shall now study an idealized and simplified model of such distributions. It seems rather appropriate to take the particles of this distribution to be stars.

Let each of the N stars belong to an n-member cluster of stars. Let each of these clusters belong to an n'-member cluster of clusters or supercluster. Finally, let the superclusters be randomly distributed in space. The stars, clusters, and superclusters form a three-level hierarchy. We will denote the position and velocity of the jth star by $\vec{y}_j$ and $\vec{z}_j$, (j=1..N), the position and velocity of the center of the kth cluster by $\vec{y}'_k$ and $\vec{z}'_k$, (k=1..N'=N/n), and the position and velocity of the center of the mth supercluster by $\vec{y}''_m$ and $\vec{z}''_m$, (m=1..N''=N'/n').

Let $P(y,z|\, y',z')$ be the conditional probability density for the distribution of stars given the distribution of clusters. Let $P'(y',z'|\, y'',z'')$ be the conditional probability density for the distribution of clusters given the distribution of superclusters. The unconditional probability

density for the stars will then be given by

$$P(\vec{r},\vec{z}) = \int P(\vec{r},\vec{z}\,|\,\vec{r}',\vec{z}')\,P(\vec{r}',\vec{z}'\,|\,\vec{r}'',\vec{z}'')\,P(\vec{r}'',\vec{z}'')\,d\vec{r}'\,d\vec{z}'\,d\vec{r}''\,d\vec{z}'' \qquad (119)$$

Let the conditional probability density for finding a particular star at $\vec{y}$ with velocity $\vec{z}$, given that the center of the cluster to which it belongs is at $\vec{y}'$ with velocity $\vec{z}'$, be $F(\vec{y}-\vec{y}',\vec{z}-\vec{z}')$. F is to be a universal cluster function and is independent of the particular star and cluster chosen. Let us assign stars 1..n to cluster 1, stars n+1..2n to cluster 2, etc.. The conditional probability density for the distribution of stars given the distribution of clusters is then

$$P(\vec{r},\vec{z}\,|\,\vec{r}'\vec{z}') = \prod_{j=1}^{N} F(\vec{Y}_j - \vec{Y}'_{k_j}, \vec{z}_j - \vec{z}'_{k_j}) \qquad (120)$$

where

$$\begin{aligned} k_j &= 1 && j = 1 \ldots n \\ k_j &= 2 && j = n+1 \ldots 2n \\ &\cdots \\ k_j &= N' && j = N-n+1 \ldots N \end{aligned} \qquad (121)$$

In precisely the same way we shall write the conditional probability density for the distribution of clusters given the distribution of superclusters as

$$P'(Y', z' | Y'', z'') = \prod_{k=1}^{N'} F'(\vec{Y}_k' - \vec{Y}_{m_k}'', \vec{z}_k' - \vec{z}_{m_k}'') \qquad (122)$$

where

$$\begin{aligned} m_k &= 1 && k = 1 \dots n' \\ &\dots \\ m_k &= N'' && k = N'-n'+1 \dots N' \end{aligned} \qquad (123)$$

and where F' is the universal supercluster function.

Finally, we assume the superclusters to be independently distributed with a probability density which depends only on their velocities.

$$P''(Y'', z'') = \frac{1}{\Omega^{N''}} \prod_{m=1}^{N''} F''(\vec{z}_m'') \qquad (124)$$

With the aid of equations (119), (120), (122), and (124) we may now construct the generating functional for this distribution.

$$\mathcal{Z}[u] = \int \frac{1}{\Omega^{N''}} \prod_{m=1}^{N''} F''(\vec{z}_m'') \prod_{k=1}^{N'} F'(\vec{Y}_k' - \vec{Y}_{m_k}'', \vec{z}_k' - \vec{z}_{m_k}'') \times \prod_{j=1}^{N} F(\vec{Y}_j - \vec{Y}_{k_j}', \vec{z}_j - \vec{z}_{k_j}') \left[1 + \tfrac{1}{2} u(\vec{r}_j, \vec{z}_j)\right] d^3Y\, d^3z\, d^3Y'\, d^3z'\, d^3Y''\, d^3z'' \qquad (125)$$

To carry out the $\vec{y}$ and $\vec{z}$ integrations note that the integrand is symmetric in those $\vec{y}_j$ and $\vec{z}_j$ corresponding to the same values of k. Similar remarks hold for the $\vec{y}'$ and $\vec{z}'$ integrations also.

$$
\begin{aligned}
\mathcal{L}[u] = \int \frac{1}{\Omega^{N''}} \prod_{m=1}^{N''} F''(\vec{z}_m'') \prod_{k=1}^{N'} F'(\vec{r}_k' - \vec{r}_{m_k}'', \vec{z}_k' - \vec{z}_{m_k}'') \\
\times \left\{ \int F(\vec{r} - \vec{r}_k', \vec{z} - \vec{z}_k') \left[1 + \tfrac{1}{2} u(\vec{r}, \vec{z}) \right] d\vec{r}\, d\vec{z} \right\}^{N} d\vec{r}_1' \cdots d\vec{r}_{N'}'\, d\vec{z}_1' \cdots d\vec{z}_{N'}'\, d\vec{r}_1'' \cdots d\vec{r}_{N''}''\, d\vec{z}_1'' \cdots d\vec{z}_{N''}'' \\
= \int \frac{1}{\Omega^{N''}} \prod_{m=1}^{N''} F''(\vec{z}_m'') \left[\int F'(\vec{r}' - \vec{r}_m'', \vec{z}' - \vec{z}_m'') \left\{ \int F(\vec{r} - \vec{r}', \vec{z} - \vec{z}') \right. \right. \\
\left. \left. \times \left[1 + \tfrac{1}{2} u(\vec{r}, \vec{z}) \right] d\vec{r}\, d\vec{z} \right\}^{N} d\vec{r}_1' \cdots d\vec{r}_{N'}'\, d\vec{z}_1' \cdots d\vec{z}_{N'}' \right]^{N'} d\vec{r}_1'' \cdots d\vec{r}_{N''}''\, d\vec{z}_1'' \cdots d\vec{z}_{N''}'' \\
= \left\{ \int \frac{1}{\Omega} F''(\vec{z}'') \left[\int F'(\vec{r}' - \vec{r}'', \vec{z}' - \vec{z}'') \left\{ \int F(\vec{r} - \vec{r}', \vec{z} - \vec{z}') \right. \right. \right. \\
\left. \left. \left. \times \left[1 + \tfrac{1}{2} u(\vec{r}, \vec{z}) \right] d\vec{r}\, d\vec{z} \right\}^{N} d\vec{r}_1' \cdots d\vec{r}_{N'}'\, d\vec{z}_1' \cdots d\vec{z}_{N'}' \right]^{N'} d\vec{r}_1'' \cdots d\vec{r}_{N''}''\, d\vec{z}_1'' \cdots d\vec{z}_{N''}'' \right\}^{N''} \\
= \left\{ \int \frac{1}{\Omega} F''(\vec{z}'') \left[1 + \int F'(\vec{r}' - \vec{r}'', \vec{z}' - \vec{z}'') \sum_{k=1}^{N} \binom{N}{k} \left\{ \frac{1}{2} \right. \right. \right. \\
\left. \left. \left. \times \int F(\vec{r} - \vec{r}', \vec{z} - \vec{z}') u(\vec{r}, \vec{z}) d\vec{r}\, d\vec{z} \right\}^{k} d\vec{r}_1' \cdots d\vec{r}_{N'}'\, d\vec{z}_1' \cdots d\vec{z}_{N'}' \right]^{N'} d\vec{r}_1'' \cdots d\vec{r}_{N''}''\, d\vec{z}_1'' \cdots d\vec{z}_{N''}'' \right\}^{N''} \\
= \left\{ 1 + \frac{1}{\Omega} \int F''(\vec{z}'') \sum_{p=1}^{N'} \binom{N'}{p} \left[\int F'(\vec{r}' - \vec{r}'', \vec{z}' - \vec{z}'') \sum_{k=1}^{N} \binom{N}{k} \right. \right. \\
\left. \left. \times \left\{ \frac{1}{2} \int F(\vec{r} - \vec{r}', \vec{z} - \vec{z}') u(\vec{r}, \vec{z}) d\vec{r}\, d\vec{z} \right\}^{k} d\vec{r}_1' \cdots d\vec{r}_{N'}'\, d\vec{z}_1' \cdots d\vec{z}_{N'}' \right]^{p} d\vec{r}_1'' \cdots d\vec{r}_{N''}''\, d\vec{z}_1'' \cdots d\vec{z}_{N''}'' \right]^{N''}
\end{aligned}
\tag{126}
$$

where we have used the notation

$$
\binom{n}{p} = \frac{n!}{p!(n-p)!} \tag{127}
$$

for the binomial coefficients.

Now, since

$$\frac{1}{\Omega} = \frac{C}{nn'N''} \tag{128}$$

and since

$$\lim_{N''\to\infty} \left(1 + \frac{x}{N''}\right)^{N''} = e^{x} \tag{129}$$

in the limit of large N'' the functional $\mathcal{L}[u]$ becomes

$$\mathcal{L}[u] = \exp\left[\frac{C}{nn'}\int F''(\vec{z}'')\sum_{p=1}^{n'}\binom{n'}{p}\left[F'(\vec{Y}'-\vec{Y}'', \vec{z}'-\vec{z}'')\right.\right.$$
$$\left.\left.\times\sum_{\ell=1}^{n}\binom{n}{\ell}\left\{\frac{1}{C}\int F(\vec{Y}-\vec{Y}', \vec{z}-\vec{z}')u(\vec{Y},\vec{z})\right\}^{\ell} d\vec{Y}'d\vec{z}'\right]^{p} d\vec{Y}''d\vec{z}''\right] \tag{130}$$

Note that this functional is exponential in form. Thus A, the generating functional for the correlation functions is simpler in form than $\mathcal{L}$, the generating functional for the distribution functions. Let us deal then with A.

$$A[u] = \frac{C}{nn'}\int F''(\vec{z}'')\sum_{p=1}^{n'}\binom{n'}{p}\left[F'(\vec{Y}'-\vec{Y}'', \vec{z}'-\vec{z}'')\right.$$
$$\left.\times\sum_{\ell=1}^{n}\binom{n}{\ell}\left\{\frac{1}{C}\int F(\vec{Y}-\vec{Y}', \vec{z}-\vec{z}')u(\vec{Y},\vec{z})\right\}^{\ell} d\vec{Y}'d\vec{z}'\right]^{p} d\vec{Y}''d\vec{z}'' \tag{131}$$

We mentioned in chapter III that if the highest order correlations involve m' particles the m-particle correlation functions vanish for m > m'. In the present case since there are exactly nn' stars per supercluster that is the largest

number of stars that can be correlated. If we examine the functional A we find it to be a polynomial of order nn'. Thus we indeed verify that for this model $g^{(p)}=0$ for $p>nn'$.

It is instructive to consider a special case of the present model, the limit of simple clustering. If we set the number of clusters per supercluster equal to one, and then demand that the cluster associated with a given supercluster be located at the supercluster center with zero relative velocity, the clusters will be independently distributed with a velocity distribution given by F". We obtain the generating functional for the correlation functions in this limit by setting n'=1 and $F'(\vec{y}',\vec{z}')=\delta(y')\,\delta(z')$ in equation (131)

$$A[u]=\frac{c}{n}\sum_{\ell=1}^{n}\binom{n}{\ell}\frac{1}{c^{\ell}}\int F''(\vec{z}'')\,F(\vec{y}_1-\vec{y}'',\vec{z}_1-\vec{z}'')\cdots F(\vec{y}_\ell-\vec{y}'',\vec{z}_\ell-\vec{z}'') \tag{132}$$
$$\times u(\vec{y}_1,\vec{z}_1)\cdots u(\vec{y}_\ell,\vec{z}_\ell)\,d\vec{y}_1\,d\vec{z}_1\cdots d\vec{y}_\ell\,d\vec{z}_\ell$$

Now, in general

$$A[u]=\sum_{\ell=1}^{\infty}\frac{1}{\ell!}\int g'^{(\ell)}(1\ldots\ell)\,u(1)\cdots u(\ell)\,d(1)\cdots d(\ell) \tag{133}$$

Comparing equations (132) and (133) we may immediately read off the correlation functions.

$$\mathcal{J}^{(n)}(1\ldots\ell) = \frac{(n-1)!}{(n-\ell)!}\, C^{1-\ell}\int F''(\vec{z}'')\,F(\vec{r}_1-\vec{r}',\vec{z}_1-\vec{z}'')\ldots F(\vec{r}_\ell-\vec{r}'',\vec{z}_\ell-\vec{z}'')\,d\vec{r}''\,d\vec{z}''$$

$$1 \le \ell \le n \qquad (134)$$

$$= 0 \qquad \ell > n$$

In the more general situation described by the full functional of equation (131) the expressions for the correlation functions are a good deal more complicated. This is because the entire summation over ℓ is raised to the pth power and the powers of u appear in a complicated way. Rather than go through the unrewarding process of deriving the general expression for the correlation functions we will be content to explicitly determine only the first two. These are in any case the most important physically. Let us therefore refer to equation (131) and extract the terms of first and second order in u. The first order term asises from the contribution of $\ell=p=1$. There are two second order terms, one from $\ell=2$, $p=1$ and one from $\ell=1$, $p=2$.

$$A[u] = \frac{c}{nn'}\int F''(\vec{z}'')\binom{n'}{1}F'(\vec{r}'-\vec{r}'',\vec{z}'-\vec{z}'')\binom{n}{1}F(\vec{r}_1-\vec{r}',\vec{z}_1-\vec{z}')\frac{1}{c}u(\vec{r}_1,\vec{z}_1)\,d\vec{r}_1\,d\vec{z}_1\,d\vec{r}'\,d\vec{z}'\,d\vec{r}''\,d\vec{z}''$$

$$+\frac{c}{nn'}\int F''(\vec{z}'')\binom{n'}{1}F'(\vec{r}'-\vec{r}'',\vec{z}'-\vec{z}'')\binom{n}{2}\frac{1}{c}F(\vec{r}_1-\vec{r}',\vec{z}_1-\vec{z}')u(\vec{r}_1,\vec{z}_1)\frac{1}{c}F(\vec{r}_2-\vec{r}',\vec{z}_2-\vec{z}')$$

$$\times u(\vec{r}_2,\vec{z}_2)\,d\vec{r}_1\,d\vec{z}_1\,d\vec{r}_2\,d\vec{z}_2\,d\vec{r}'\,d\vec{z}'\,d\vec{r}''\,d\vec{z}'' \qquad (135)$$

$$+\frac{c}{nn'}\int F''(\vec{z}'')\binom{n'}{2}F'(\vec{r}_1'-\vec{r}'',\vec{z}_1'-\vec{z}'')\binom{n}{1}\frac{1}{c}F(\vec{r}_1-\vec{r}_1',\vec{z}_1-\vec{z}_1')u(\vec{r}_1,\vec{z}_1)$$

$$\times F'(\vec{r}_2'-\vec{r}'',\vec{z}_2'-\vec{z}'')\binom{n}{1}\frac{1}{c}F(\vec{r}_2-\vec{r}_2',\vec{z}_2-\vec{z}_2')\,d\vec{r}_1\,d\vec{z}_1\,d\vec{r}_2\,d\vec{z}_2\,d\vec{r}_1'\,d\vec{z}_1'\,d\vec{r}_2'\,d\vec{z}_2'\,d\vec{r}''\,d\vec{z}''$$

$$+ O(u^3)$$

We may read off the first two correlation functions by comparing equations (133) and (135).

$$\mathcal{G}^{(1)}(\vec{r}_1,\vec{z}_1) = \int F''(\vec{z}'')\,F'(\vec{r}'-\vec{r}'',\vec{z}'-\vec{z}'')\,F(\vec{r}_1-\vec{r}',\vec{z}_1-\vec{z}')\,d\vec{r}'\,d\vec{z}'\,d\vec{r}''\,d\vec{z}''$$

$$\mathcal{G}^{(2)}(\vec{r}_1,\vec{z}_1,\vec{r}_2,\vec{z}_2) = \frac{n-1}{c}\int F''(\vec{z}'')\,F'(\vec{r}'-\vec{r}'',\vec{z}'-\vec{z}'')\,F(\vec{r}_1-\vec{r}',\vec{z}_1-\vec{z}')\,F(\vec{r}_2-\vec{r}',\vec{z}_2-\vec{z}')\,d\vec{r}'\,d\vec{z}'\,d\vec{r}''\,d\vec{z}''$$

$$+\frac{n(n-1)}{c}\int F''(\vec{z}'')\,F'(\vec{r}_1'-\vec{r}'',\vec{z}_1'-\vec{z}'')\,F(\vec{r}_1'-\vec{r}_1,\vec{z}_1'-\vec{z}_1)\,F'(\vec{r}_2'-\vec{r}'',\vec{z}_2'-\vec{z}'')\,F(\vec{r}_2-\vec{r}_2',\vec{z}_2-\vec{z}_2')\,d\vec{r}_1'\,d\vec{z}_1'\,d\vec{r}_2'\,d\vec{z}_2'\,d\vec{r}''\,d\vec{z}'' \qquad (136)$$

We have seen that S(k), the Fourier transform of the velocity integrated two-particle correlation function, describes the distribution of peculiar binding energy in Fourier space. Let us calculate this quantity for the present model.

To begin we define two quantities.

$$\mathcal{F}(\vec{r}) = \int F(\vec{r},\vec{z})\,d\vec{z}$$

$$\mathcal{F}'(\vec{r}') = \int F(\vec{r}',\vec{z}')\,d\vec{z}' \qquad (137)$$

$\mathcal{F}(\vec{y})$ is essentially the average density profile of a cluster and $\mathcal{F}'(\vec{y}')$ is the corresponding supercluster quantity. The velocity integrated two-particle correlation function is easily expressed in terms of these density profiles.

$$\int g^{(2)}(\vec{r}_1,\vec{z}_1,\vec{r}_2,\vec{z}_2)\,d\vec{z}_1\,d\vec{z}_2 = \frac{n-1}{c}\int \mathcal{F}(\vec{r}_1-\vec{r}')\,\mathcal{F}(\vec{r}_2-\vec{r}')\,d\vec{r}'$$

$$+\frac{n(n'-1)}{c}\int \mathcal{F}(\vec{r}_1'-\vec{r}'')\,\mathcal{F}(\vec{r}_2'-\vec{r}'')\,\mathcal{F}(\vec{r}_1-\vec{r}_1')\,\mathcal{F}(\vec{r}_2-\vec{r}_2')\,d\vec{r}_1'\,d\vec{r}_2'\,d\vec{r}'' \qquad (138)$$

Writing this equation in terms of the difference coordinates $\vec{y}=\vec{y}_1-\vec{y}_2$ and then taking the Fourier transform with respect to $\vec{y}$ we obtain

$$S(\vec{x}) = \frac{n-1}{c}\int e^{-2\pi i\vec{x}\cdot\vec{r}}\,\mathcal{F}(\vec{r}-\vec{r}')\,\mathcal{F}(-\vec{r}')\,d\vec{r}\,d\vec{r}'$$

$$+\frac{n(n'-1)}{c}\int e^{-2\pi i\vec{x}\cdot\vec{r}}\,\mathcal{F}(\vec{r}_1'-\vec{r}'')\,\mathcal{F}(\vec{r}_2'-\vec{r}'')\,\mathcal{F}(\vec{r}-\vec{r}_1')\,\mathcal{F}(-\vec{r}_2')\,d\vec{r}\,d\vec{r}_1'\,d\vec{r}_2'\,d\vec{r}'' \qquad (139)$$

$$= \frac{n-1}{c}\,\bar{\mathcal{F}}(\vec{x})\,\bar{\mathcal{F}}(-\vec{x}) + \frac{n(n'-1)}{c}\,\bar{\mathcal{F}}(\vec{x})\,\bar{\mathcal{F}}(-\vec{x})\,\bar{\mathcal{F}}(\vec{x})\,\bar{\mathcal{F}}(-\vec{x})$$

$$= \frac{n-1}{c}\,|\bar{\mathcal{F}}(\vec{x})|^2 + \frac{n(n'-1)}{c}\,|\bar{\mathcal{F}}(\vec{x})|^2\,|\bar{\mathcal{F}}'(\vec{x})|^2$$

The last step follows from the reality of the profile functions.

To this point the cluster and supercluster shapes have been left arbitrary. For the sake of definiteness let us now take them to be spherically symmetric with Gaussian radial dependences.

$$\mathcal{F}(\vec{r}) = \text{Const.}\; e^{-\frac{r^2}{2R_c^2}}$$

$$\mathcal{F}'(\vec{r}') = \text{Const.}\; e^{-\frac{r'^2}{2R_s^2}} \qquad (140)$$

where the constants are to be chosen so that $\mathcal{F}$ and $\mathcal{F}'$ integrate to unity. λ_c is the characteristic cluster dimension and λ_s is the corresponding supercluster quantity. The Fourier transforms of these shape functions are

$$\bar{\mathcal{F}}(\ell) = e^{-\ell^2\lambda_c^2/2}$$

$$\bar{\mathcal{F}}'(\ell) = e^{-\ell^2\lambda_s^2/2} \tag{141}$$

Substituting equations (141) into equation (139) one obtains

$$S(\ell) = \frac{n-1}{c} e^{-\ell^2\lambda_c^2} + \frac{n(n'-1)}{c} e^{-\ell^2(\lambda_s^2+\lambda_c^2)} \tag{142}$$

This function is graphically illustrated on the next page. Note that it is not characterized by peaks at $k=1/\lambda_c$ nd $k=1/\lambda_s$. Thus, even a precisely hierarchical distribution is not necessarily characterized by a sharply peaked clustering spectrum.

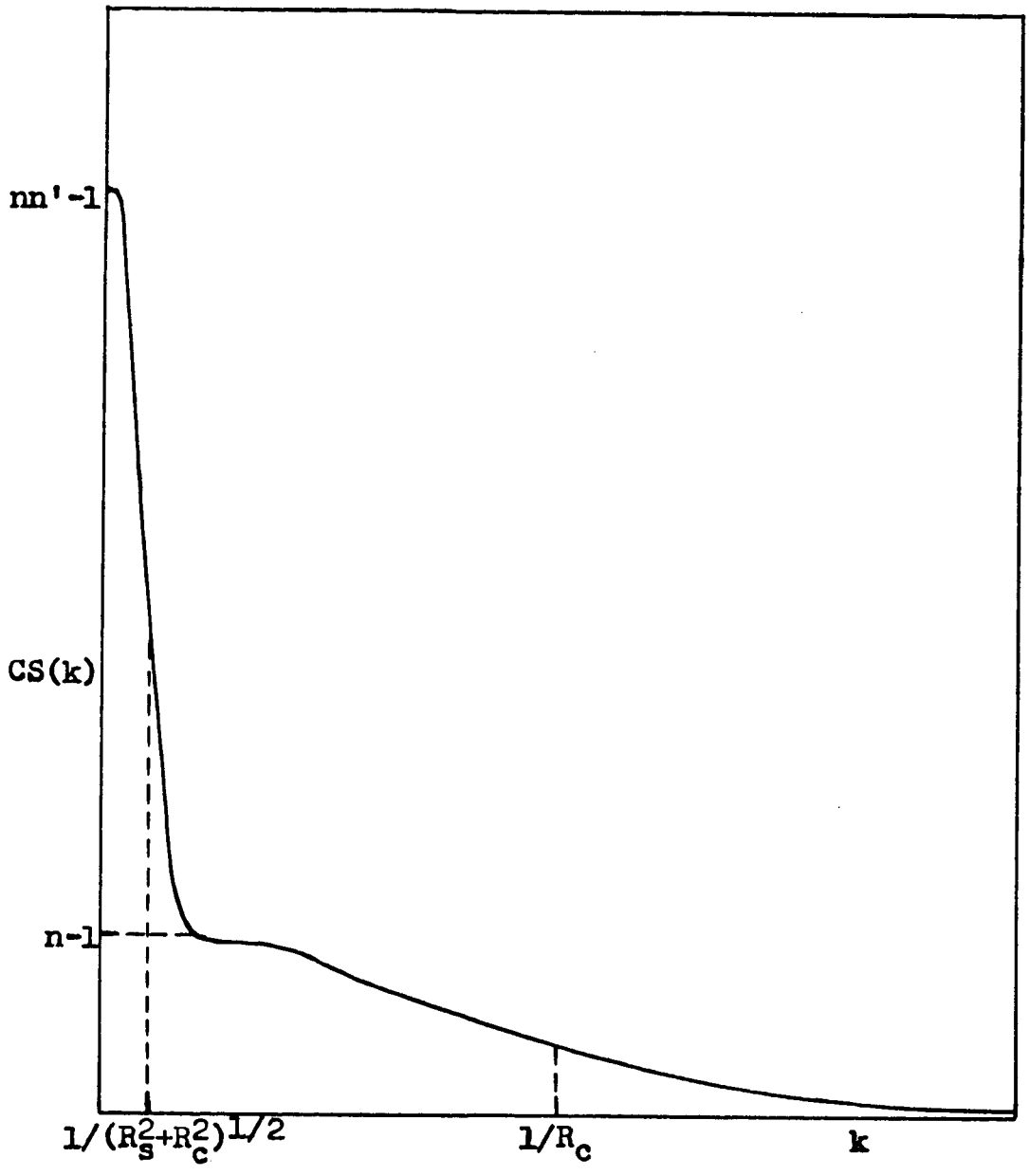


FIGURE 2

THE CLUSTERING SPECTRUM S(k)

FOR A THREE-LEVEL HIERARCHICAL DISTRIBUTION

CHAPTER V

THE DYNAMICS OF CLUSTERING

Up to this point we have said nothing about the solutions of the equations of motion for the correlation functions. Recall that these equations are

$$\frac{\partial g^{(n)}}{\partial s} + \sum_{j=1}^{n} \vec{z}_j \cdot \frac{\partial g^{(n)}}{\partial \vec{r}_j} + \sum_{j,k=1}^{n}{}' \vec{F}_{j,k} \cdot \frac{\partial}{\partial \vec{z}_j} \left[g^{(n)}(1\ldots n) + \sum_{p}{}' g^{(n)}(\ldots j\ldots) g^{(n-n)}(\ldots k\ldots) \right]$$
$$+ C \sum_{j=1}^{n} \int \vec{F}_{j,n+1} \cdot \frac{\partial}{\partial \vec{z}_j} \left[g^{(n+1)}(1\ldots n+1) + \sum_{p}{}' g^{(n)}(\ldots j\ldots) g^{(n+1-n)}(\ldots n+1) \right] d(n+1) = 0 \tag{143}$$

where

$$\vec{F}_{j,k} = -MG\,R(s)\,\frac{\vec{r}_j - \vec{r}_k}{|\vec{r}_j - \vec{r}_k|^3} \tag{144}$$

These are an infinite set of coupled, non-linear, integro-differential equations. Although their general solution is clearly hopeless we may derive from the first two of them an exact conservation relation. We shall also investigate these equations in the limit of weak clustering and, in that limit, obtain a useful integral equation for the clustering spectrum.

I. THE ENERGY EQUATION

Let us recall the expressions for the specific peculiar kinetic energy T_m, and the specific peculiar binding energy U_m.

$$T_m = \frac{R(s)^{-2}}{2} \int g^{(1)}(\vec{z}_1) z_1^2 \, d\vec{z}_1 \tag{145}$$

and

$$U_m = -\frac{MCG\, R(s)^{-1}}{2} \int g^{(2)}(\vec{Y}; \vec{z}_1, \vec{z}_2) \frac{1}{Y} d\vec{z}_1 \, d\vec{z}_2 \, d\vec{Y} \tag{146}$$

We shall now derive a conservation equation for these quantities.

Let us first explicitly write down the first two of equations (143) making use of the information that $g^{(1)}(\vec{y}_1, \vec{z}_1)$ is actually independent of $\vec{y}_1$, that $g^{(2)}(\vec{y}_1 \vec{z}_1, \vec{y}_2 \vec{z}_2)$ depends on the coordinates $\vec{y}_1$ and $\vec{y}_2$ only through their difference $\vec{y}$, and that $\vec{F}_{j,k} = -\vec{F}_{k,j}$.

$$\frac{\partial g^{(1)}(\vec{z}_1)}{\partial s} + c \int \vec{F}_{1,2} \cdot \frac{\partial}{\partial \vec{z}_1} g^{(2)}(\vec{Y}; \vec{z}_1, \vec{z}_2) \, d\vec{z}_2 \, d\vec{Y} = 0 \tag{147}$$

$$\frac{\partial}{\partial s} g^{(2)}(\vec{Y}; \vec{z}_1, \vec{z}_2) + (\vec{z}_1 - \vec{z}_2) \cdot \frac{\partial}{\partial \vec{Y}} g^{(2)}(\vec{Y}; \vec{z}_1, \vec{z}_2) + \vec{F}_{1,2} \cdot \left(\frac{\partial}{\partial \vec{z}_1} - \frac{\partial}{\partial \vec{z}_2}\right)\left(g^{(2)}(1,2) + g^{(1)}(1) g^{(1)}(2)\right)$$

$$+ c \int \vec{F}_{1,3} \cdot \frac{\partial}{\partial \vec{z}_1} \left[g^{(3)}(1,2,3) + g^{(2)}(2,3) g^{(1)}(1) \right] d(3) \tag{148}$$

$$+ c \int \vec{F}_{2,3} \cdot \frac{\partial}{\partial \vec{z}_2} \left[g^{(3)}(1,2,3) + g^{(2)}(1,3) \, g^{(1)}(2) \right] d(3) = 0$$

We now multiply equation (147) by $z_1^2/2$ and integrate over $\vec{z}_1$. The second term is to be integrated by parts and the surface term discarded since we assume $g^{(2)}$ vanishes rapidly at large velocities.

$$\frac{\partial}{\partial s}\int g^{(1)}(\vec{z}_1)\frac{z_1^2}{2}\,d\vec{z}_1 + MGCR\int\frac{\vec{Y}\cdot\vec{z}_1}{Y^3}\,g^{(2)}(\vec{Y};\vec{z}_1,\vec{z}_2)\,d\vec{z}_1 d\vec{z}_2\,d\vec{Y} = 0 \qquad (149)$$

Next we divide equation (148) by 2y and integrate over $\vec{y}$, $\vec{z}_1$, and $\vec{z}_2$. Here the $\vec{y}$ integration is done by parts also. Again we neglect the surface term because $g^{(2)}$ is assumed to vanish strongly with increasing y.

$$\frac{\partial}{\partial s}\int g^{(2)}(\vec{Y};\vec{z}_1,\vec{z}_2)\frac{1}{2Y}\,d\vec{z}_1 d\vec{z}_2\,d\vec{Y} + \int\frac{(\vec{z}_1-\vec{z}_2)\cdot\vec{Y}}{2Y^3}\,g^{(2)}(\vec{Y};\vec{z}_1,\vec{z}_2)\,d\vec{z}_1 d\vec{z}_2\,d\vec{Y} = 0 \qquad (150)$$

Since $g^{(2)}$ is symmetric with respect to particle indices 1 and 2

$$g^{(2)}(\vec{Y};\vec{z}_1,\vec{z}_2) = g^{(2)}(-\vec{Y};\vec{z}_2,\vec{z}_1) \qquad (151)$$

Equation (150) may thus be rewritten as

$$\frac{\partial}{\partial s}\int g^{(2)}(\vec{Y};\vec{z}_1,\vec{z}_2)\frac{1}{2Y}\,d\vec{z}_1 d\vec{z}_2 d\vec{Y} + \int\frac{\vec{z}_1\cdot\vec{Y}}{Y^3}\,g^{(2)}(\vec{Y};\vec{z}_1,\vec{z}_2)\,d\vec{z}_1 d\vec{z}_2 d\vec{Y} = 0 \qquad (152)$$

We now multiply equation (152) by MCGR and subtract it from equation (149).

$$\frac{\partial}{\partial s}\int g^{(1)}(1)\frac{z_1^2}{2}\,dz_1 - \frac{MCGR}{2}\frac{\partial}{\partial s}\int g^{(2)}(\vec{Y};\vec{z}_1,\vec{z}_2)\frac{1}{Y}\,d\vec{z}_1 d\vec{z}_2 d\vec{Y} = 0 \qquad (153)$$

Or

$$\frac{d}{ds}\,(R^2 T_m) + R\,\frac{d}{ds}(R U_m) = 0 \qquad (154)$$

Now the time-like variable s is related to physical time t through equations (15).

$$\frac{d}{ds} = R^2 \frac{d}{dt} \tag{155}$$

Thus

$$R^2 \frac{d}{dt}(R^2 T_m) + R^3 \frac{d}{dt}(R U_m) = 0 \tag{156}$$

Or

$$\frac{d}{dt}(T_m + U_m) + H(2T_m + U_m) = 0 \tag{157}$$

Where

$$H = \frac{1}{R} \frac{dR}{dt} \tag{158}$$

Equation (157) is the energy theorem first derived by Irvine (4) for a continuous distribution and in terms of Layzer's peculiar binding energy. One of the most significant features of this equation is that it is exact. It is probably the only exact result obtainable from equations of motion (143).

II. THE WEAK CLUSTERING APPROXIMATION

In order to proceed further with the solution of equations (143) we shall make two simplifying assumptions. The first of these is that the range of correlations is sufficiently great compared to the mean interparticle distance that the particulate nature of the distribution

does not appreciably affect the dynamics of clustering. In other words we shall assume the applicability of the continuum limit. The second assumption is that the intensity of clustering is weak and that a linear analysis may therefore be employed.

Suppose the clustering is characterized by a correlation range λ and an intensity ϵ. ϵ may be taken to be the contrast of density fluctuations in the more usual continuum formulation. The continuum limit of equations (143) may be expected to apply when the force on a given particle due to its near neighbors, i.e. particles at a distance of order $C^{-1/3}$, is much less than the force due to the density fluctuations. Thus, the condition for the applicability of the continuum limit is

$$\frac{MGR}{(C^{-1/3})^2} \ll \frac{(\epsilon MC\lambda^3)GR}{\lambda^2} \tag{159}$$

Or

$$C^{-1/3} \ll \lambda\epsilon \tag{160}$$

Since we shall assume the clustering to be weak, and therefore ϵ to be small, condition (160) is stronger that the mere requirement that there be many particles within a sphere of radius λ.

As the clustering progresses ϵ will increase and most probably λ will also. Therefore if inequality (160) is initially satisfied it will be satisfied for later times too. We may thus regard (160) as a condition on the initial distribution.

To obtain the continuum limit of equations (143) we merely discard those terms involving $\vec{F}_{j,k}$, and therefore M, without a compensating factor C. Thus, in the continuum limit, equations (143) become

$$\frac{\partial g^{(n)}}{\partial s} + \sum_{j=1}^{n} \vec{z}_j \cdot \frac{\partial g^{(n)}}{\partial \vec{r}_j} + C \sum_{j=1}^{n} \int \vec{F}_{j,n+1} \cdot \frac{\partial}{\partial \vec{z}_j} \left[g^{(n+1)}(1 \ldots n+1) + \sum{}' g^{(n)}(\ldots j \ldots) g^{(n+1-N)}(\ldots n+1) \right] d(n+1) = 0 \tag{161}$$

We shall express the second assumption, that the clustering is weak, by demanding that for $n \geq 2$ $g^{(n)}$ be of order n in ϵ . We may motivate this particular assignment of order in ϵ as follows.

It seems reasonable that, at least as far as the orders of magnitude of the various correlations are concerned, the weakly clustered distribution presently under consideration will be similar to a generalized Poisson distribution of chapter IV. Such a distribution may be said

to be weakly clustered if $\tilde{p}$, the fluctuating part of the random field p, is small compared to $\bar{p}$, the average part. We express this by writing

$$P(\vec{Y},\vec{z}) = \bar{P}(\vec{Y},\vec{z}) + \epsilon\, \tilde{P}(\vec{Y},\vec{z}) \tag{162}$$

Now, the generating functional for the generalized Poisson distribution is given by equation (109) as

$$\mathcal{L}[u] = \left\langle e^{\int P(\vec{Y},\vec{z})\, u(\vec{Y},\vec{z})\, d\vec{Y}\, d\vec{z}} \right\rangle \tag{163}$$
$$= e^{\int \bar{P}(\vec{z})\, u(\vec{Y},\vec{z})\, d\vec{Y}\, d\vec{z}} \left\langle e^{\epsilon \int \tilde{P}(\vec{Y},\vec{z})\, u(\vec{Y},\vec{z})\, d\vec{Y}\, d\vec{z}} \right\rangle$$

Let us define a new functional A' by

$$e^{A'[\epsilon u]} = \left\langle e^{\epsilon \int \tilde{P}(\vec{Y},\vec{z})\, u(\vec{Y},\vec{z})\, d\vec{Y}\, d\vec{z}} \right\rangle \tag{164}$$

A' presumably has a power series expansion. Therefore

$$A'[\epsilon u] = \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \int \frac{\delta^n A'[u]}{\delta u(1) \ldots \delta u(n)} \Big]_{u=0} u(1) \ldots u(n)\, d(1) \ldots d(n) \tag{165}$$

where the only dependence on ϵ is shown explicitly.

The first term of this series must actually vanish since by equations (112), (163), and (164)

$$\bar{P}(\vec{z}) = S^{(1)}(\vec{z})$$
$$= \frac{\delta \mathcal{L}}{\delta u(1)} \Big]_{u=0} \tag{166}$$

$$= \bar{P}(\vec{z}) + \left[e^{A'[\epsilon u]} \epsilon \frac{\delta A'}{\delta u(1)} \right]_{u=0}$$

Or

$$\frac{\delta A'}{\delta u(1)} \Big]_{u=0} = 0 \tag{167}$$

We may therefore write the generating functional $\mathcal{L}$ as

$$\mathcal{L}[u] = e^{\int \bar{P}(\vec{z}) u(\vec{r},\vec{s}) d\vec{r} d\vec{s} + \sum_{n=2}^{\infty} \frac{\epsilon^n}{n!} \int \frac{\delta^n A'}{\delta u(1) \ldots \delta u(n)} \Big]_{u=0} u(1) \ldots u(n) d(1) \ldots d(n)} \tag{168}$$

We see at once that the correlation functions are given by

$$\begin{aligned} g^{(1)}(1) &= \bar{P}(\vec{z}_1) \\ g^{(n)}(1 \ldots n) &= \epsilon^n \frac{\delta^n A'}{\delta u(1) \ldots \delta u(n)} \end{aligned} \tag{169}$$

Since the only dependence on ϵ is shown explicitly the order in ϵ of the various correlations is as assumed.

This assignment of order in ϵ will be meaningful only if it is maintained by the equations of motion. It is necessary that the equation involving $\frac{\partial g^{(n)}}{\partial s}$ have no terms of order $\epsilon^{n'}$ where $n' < n$. If this is not the case $g^{(n)}$ will quickly grow to order $\epsilon^{n'}$. Reference to equations (161) shows that in the continuum limit the equations for the correlation functions do indeed satisfy this condition. This is in sharp distinction to the more general set of equations (143) Here for example the equation for $g^{(2)}(1,2)$ involves

a term

$$\vec{F}_{1,2} \cdot \left(\frac{\partial}{\partial \vec{z}_1} - \frac{\partial}{\partial \vec{z}_2} \right) g^{(1)}(1)\, g^{(1)}(2)$$

which is of the zeroth order in ϵ. Thus, even if $g^{(2)}$ is initially of order ϵ^2 it will quickly grow to order unity.

Now, consistent with the assumption of weak clustering, we shall agree to retain terms of at most second order in ϵ. In that case we are left with only the first two of equations (161). Written out explicitly these surviving equations are

$$\frac{\partial g^{(1)}(\vec{z}_1)}{\partial s} + c \int \vec{F}_{1,2} \cdot \frac{\partial}{\partial \vec{z}_1}\, g^{(2)}(\vec{Y}; \vec{z}_1, \vec{z}_2)\, d\vec{z}_2\, d\vec{Y} = 0 \tag{170}$$

and

$$\begin{aligned} &\frac{\partial g^{(2)}(\vec{Y}; \vec{z}_1, \vec{z}_2)}{\partial s} + (\vec{z}_1 - \vec{z}_2) \cdot \frac{\partial}{\partial \vec{Y}}\, g^{(2)}(\vec{Y}; \vec{z}_1, \vec{z}_2) \\ &\quad + c \int \vec{F}_{1,3} \cdot \frac{\partial}{\partial \vec{z}_1}\, g^{(2)}(2,3)\, g^{(1)}(1)\, d(3) \\ &\quad + c \int \vec{F}_{2,3} \cdot \frac{\partial}{\partial \vec{z}_2}\, g^{(2)}(1,3)\, g^{(1)}(2)\, d(3) = 0 \end{aligned} \tag{171}$$

Since the time derivative of $g^{(1)}$ is of second order in ϵ we express $g^{(1)}$ as

$$g^{(1)}(\vec{z}_1, s) = \phi(\vec{z}_1) + g_2^{(1)}(\vec{z}_1, s) \tag{172}$$

where $\phi(\vec{z})$ is the zeroth order velocity distribution func-

tion and $g_2^{(1)}$ is the second order correction. $\phi(\vec{z})$ is independent of the time and is determined solely by the initial conditions. Substituting equation (172) into equations (170) and (171) we obtain

$$\frac{\partial \phi(\vec{z}_1)}{\partial s} + C \int \vec{F}_{1,2} \cdot \frac{\partial}{\partial \vec{z}_1} g^{(2)}(\vec{r}; \vec{z}_1, \vec{z}_2)\, d\vec{z}_2\, d\vec{r} = 0 \tag{173}$$

and

$$\begin{aligned} &\frac{\partial g^{(2)}(\vec{r}; \vec{z}_1, \vec{z}_2)}{\partial s} + (\vec{z}_1 - \vec{z}_2) \cdot \frac{\partial}{\partial \vec{r}} g^{(2)}(\vec{r}; \vec{z}_1, \vec{z}_2) \\ &\quad + C \frac{\partial \phi(\vec{z}_1)}{\partial \vec{z}_1} \cdot \int \vec{F}_{1,3}\, g^{(2)}(2,3)\, d(3) \\ &\quad + C \frac{\partial \phi(\vec{z}_2)}{\partial \vec{z}_2} \int \vec{F}_{2,3}\, g^{(2)}(1,3)\, d(3) = 0 \end{aligned} \tag{174}$$

These equations are more conveniently expressed in terms of the spatial Fourier transform of $g^{(2)}$.

$$\bar{g}^{(2)}(\vec{x}; \vec{z}_1, \vec{z}_2) = \int e^{-2\pi i \vec{x} \cdot \vec{r}} g^{(2)}(\vec{r}; \vec{z}_1, \vec{z}_2)\, d\vec{r} \tag{175}$$

Recalling that

$$\begin{aligned} \vec{F}_{1,2} &= -MG R(s) \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} \\ &= MG R \frac{\partial}{\partial \vec{r}_1} \int e^{2\pi i \vec{x} \cdot (\vec{r}_1 - \vec{r}_2)} \frac{1}{\pi x^2}\, d\vec{x} \\ &= i\, 2MG R \int e^{2\pi i \vec{x} \cdot (\vec{r}_1 - \vec{r}_2)} \frac{\vec{x}}{x^2}\, d\vec{x} \end{aligned} \tag{176}$$

and again letting

$$\omega^2 = MCG4\pi \tag{177}$$

we substitute equations (175) and (176) into (173) to obtain

$$\frac{\partial \bar{g}_2^{(1)}(\vec{z}_1)}{\partial s} - i2\frac{\omega^2}{4\pi}R\frac{\partial}{\partial \vec{z}_1}\cdot\int\frac{\vec{\kappa}}{\kappa^2}\bar{g}^{(2)}(\vec{\kappa};\vec{z}_1,\vec{z}_2)\,d\vec{z}_2\,d\vec{\kappa} = 0 \qquad (178)$$

We may also take the fourier transform of equation (174) using the symmetry relation (151) to obtain

$$\begin{aligned}\frac{\partial \bar{g}^{(2)}(\vec{\kappa};\vec{z}_1,\vec{z}_2)}{\partial s} &+ 2\pi i\,\vec{\kappa}\cdot(\vec{z}_1-\vec{z}_2)\bar{g}^{(2)}(\vec{\kappa};\vec{z}_1,\vec{z}_2)\\ &+ i2\frac{\omega^2}{4\pi}R\frac{\partial\phi(\vec{z}_1)}{\partial\vec{z}_1}\cdot\frac{\vec{\kappa}}{\kappa^2}\int\bar{g}^{(2)}(\vec{\kappa};\vec{z}_3,\vec{z}_2)\,d\vec{z}_3 \qquad (179)\\ &+ i2\frac{\omega^2}{4\pi}R\frac{\partial\phi(\vec{z}_2)}{\partial\vec{z}_2}\cdot\frac{\vec{\kappa}}{\kappa^2}\int\bar{g}^{(2)}(\vec{\kappa};\vec{z}_1,\vec{z}_3)\,d\vec{z}_3 = 0\end{aligned}$$

It was mentioned in the introduction that Layzer's principal conjectures concerning the development of the clustering spectrum were that highly developed regions of this spectrum tend to grow at the expense of less highly developed regions, and also that there is a general flow of binding energy from the large wave number end of the spectrum to the small wave number end. We notice however that k appears in equation (179) only as a parameter. Thus, to the accuracy of this linear analysis, the various parts of the clustering spectrum are independent. We therefore conclude that the flow of binding energy envisaged by Layzer is an essentially non-linear phenomenon and lies beyond the scope of this linear theory. One would hope to be able to treat this energy flow in the lowest approximation by retaining in equations (143) terms of order ϵ^3 and ϵ^4 too.

Such an analysis will be quite complicated and will not be attempted in this thesis. Rather, we shall continue with the linear theory, partly because the initial development of the clustering spectrum is itself of considerable interest, and partly because an understanding of the linear theory is a clear prerequisite of any non-linear treatment.

III. AN INTEGRAL EQUATION FOR THE CLUSTERING SPECTRUM

In order to derive a simple integral equation for the clustering spectrum it will prove useful to consider only those solutions of equation (179) which develop from initial conditions of the form

$$\bar{g}^{(2)}(\vec{k};\vec{z},\vec{z}_2)\Big]_{s=0} = h(\vec{k},\vec{z}_1)\,h^{*}(\vec{k},\vec{z}_2)\,H(\vec{k}) \qquad (180)$$

Here h and H are arbitrary functions having appropriate symmetry. These initial conditions include as a special case distributions which are correlated in position only. Since the restriction applies only to the initial state and involves no assumption as to the time dependence of the correlations it appears that no serious loss of generality results.

For the sake of clarity we shall henceforth explicitly indicate all time dependence.

Let us define

$$\tilde{\mathcal{G}}(\vec{\mathcal{R}};\vec{z},\vec{z}_2,s) = e^{2\pi i \vec{\mathcal{R}}\cdot(\vec{z}_1-\vec{z}_2)s}\,\bar{\mathcal{G}}^{(e)}(\vec{\mathcal{R}};\vec{z},\vec{z}_2,s) \tag{181}$$

Substituting this definition into equation (179) we find

$$\begin{aligned}&\frac{\partial\tilde{\mathcal{G}}(\vec{\mathcal{R}};\vec{z},\vec{z}_2,s)}{\partial s}\\ &\quad + i\frac{\omega^2 R(s)}{2\pi}e^{2\pi i\vec{\mathcal{R}}\cdot\vec{z}_1 s}\frac{\partial\phi(\vec{z}_1)}{\partial\vec{z}_1}\cdot\frac{\vec{\mathcal{R}}}{\mathcal{R}^2}\int e^{-2\pi i\vec{\mathcal{R}}\cdot\vec{z}_3 s}\tilde{\mathcal{G}}(\vec{\mathcal{R}};\vec{z}_3\vec{z}_2,s)\,d\vec{z}_3 \qquad (182)\\ &\quad - i\frac{\omega^2 R(s)}{2\pi}e^{-2\pi i\vec{\mathcal{R}}\cdot\vec{z}_2 s}\frac{\partial\phi(\vec{z}_2)}{\partial\vec{z}_2}\cdot\frac{\vec{\mathcal{R}}}{\mathcal{R}^2}\int e^{2\pi i\vec{\mathcal{R}}\cdot\vec{z}_3 s}\tilde{\mathcal{G}}(\vec{\mathcal{R}};\vec{z}_1,\vec{z}_3,s)\,d\vec{z}_3 = 0\end{aligned}$$

Consistent with the assumed initial conditions let us seek solutions of the form

$$\tilde{\mathcal{G}}(\vec{\mathcal{R}};\vec{z},\vec{z}_2,s) = h(\vec{\mathcal{R}},\vec{z}_1,,s)\,h^*(\vec{\mathcal{R}},\vec{z}_2,s)\,H(\vec{\mathcal{R}}) \tag{183}$$

where $H(\vec{k})$ is time independent. Substituting equation (183) into equation (182) we find

$$\begin{aligned}&h^*(\vec{\mathcal{R}},\vec{z}_2,s)\left[\frac{\partial h(\mathcal{R},z_1,s)}{\partial s} + i\frac{\omega^2 R}{2\pi}e^{2\pi i\vec{\mathcal{R}}\cdot\vec{z}_1 s}\frac{\partial\phi(\vec{z}_1)}{\partial\vec{z}_1}\cdot\frac{\vec{\mathcal{R}}}{\mathcal{R}^2}\int e^{-2\pi i\vec{\mathcal{R}}\cdot\vec{z}_3 s}h(\vec{\mathcal{R}},\vec{z}_3,s)\,d\vec{z}_3\right] \qquad (184)\\ &+h(\vec{\mathcal{R}},\vec{z}_1,,s)\left[\frac{\partial h^*(\vec{\mathcal{R}},\vec{z}_2,s)}{\partial s} - i\frac{\omega^2 R}{2\pi}e^{-2\pi i\vec{\mathcal{R}}\cdot\vec{z}_2 s}\frac{\partial\phi(\vec{z}_2)}{\partial\vec{z}_2}\cdot\frac{\vec{\mathcal{R}}}{\mathcal{R}^2}\int e^{2\pi i\vec{\mathcal{R}}\cdot\vec{z}_3 s}h^*(\mathcal{R},\vec{z}_3,s)\,d\vec{z}_3\right] = 0\end{aligned}$$

We find that a solution does result provided h satisfies the equation

$$\frac{\partial h(\vec{\mathcal{R}},\vec{z},s)}{\partial s} + \frac{i\omega^2 R}{2\pi}e^{2\pi i\vec{\mathcal{R}}\cdot\vec{z}s}\frac{\partial\phi(z)}{\partial\vec{z}}\cdot\frac{\vec{\mathcal{R}}}{\mathcal{R}^2}\int e^{-2\pi i\vec{\mathcal{R}}\cdot\vec{z}'s}h(\vec{\mathcal{R}},\vec{z}',s)\,d\vec{z}' = 0 \tag{185}$$

We may obtain an integral equation for h by integrat-

ing equation (185) with respect to s.

$$h(\vec{k},\vec{z},s) + \frac{i\omega^2}{2\pi}\int_0^s R(s')e^{2\pi i \vec{k}\cdot\vec{z}s'}\frac{\partial\phi(\vec{z})}{\partial\vec{z}}\cdot\frac{\vec{k}}{k^2}\int e^{-2\pi i\vec{k}\cdot\vec{z}'s'}h(\vec{k},\vec{z}',s')\,d\vec{z}'\,ds'$$
$$= h(\vec{k},\vec{z},0) \tag{186}$$

Here $h(\vec{k},\vec{z},0)$ is determined by the initial conditions.

Now the quantity of principal interest is the clustering spectrum S(k).

$$\begin{aligned} S(k,s) &= \int \bar{g}^{(2)}(\vec{k},\vec{z},\vec{z}_2,s)\,d\vec{z}_1\,d\vec{z}_2 \\ &= \int e^{-2\pi i\vec{k}\cdot(\vec{z}_1-\vec{z}_2)s}h(\vec{k},\vec{z}_1,s)h^*(\vec{k},\vec{z}_2,s)H(\vec{k})\,d\vec{z}_1\,d\vec{z}_2 \\ &= H(\vec{k})\left|\int e^{-2\pi i\vec{k}\cdot\vec{z}s}h(\vec{k},\vec{z},s)\,d\vec{z}\right|^2 \end{aligned} \tag{187}$$

Let

$$\Im(\vec{k},s) = \int e^{-2\pi i\vec{k}\cdot\vec{z}s}h(\vec{k},\vec{z},s)\,d\vec{z} \tag{188}$$

To obtain an integral equation for $\Im$ we multiply equation (186) by $e^{-2\pi i\vec{k}\cdot\vec{z}s}$ and integrate with respect to $\vec{z}$.

$$\Im(\vec{k},s) + \frac{i\omega^2}{2\pi}\int_0^s R(s')\int e^{-2\pi i\vec{k}\cdot\vec{z}(s-s')}\frac{\partial\phi(\vec{z})}{\partial\vec{z}}\cdot\frac{\vec{k}}{k^2}\,\Im(\vec{k},s')\,d\vec{z}\,ds'$$
$$= \int e^{-2\pi i\vec{k}\cdot\vec{z}s}h(\vec{k},\vec{z},0)\,d\vec{z} \tag{189}$$

Let $\vec{\phi}(\vec{p})$ represent the velocity Fourier transform of

$\phi(\vec{z})$.

$$\bar{\phi}(\vec{p}) = \int e^{-2\pi i \vec{p}\cdot\vec{a}} \phi(\vec{a})\, d\vec{a} \tag{190}$$

The velocity integration on the left hand side of equation (189) may now be done by parts.

$$\begin{aligned} \zeta(\vec{x},s) - \omega^2 \int_0^s R(s')\bar{\phi}[\vec{x}(s-s')](s-s')\,\zeta(\vec{x},s')\,ds' \\ = \int e^{-2\pi i \vec{x}\cdot\vec{z}s} f(\vec{x},\vec{z},0)\, d\vec{z} \end{aligned} \tag{191}$$

The quantities $\bar{\phi}(\vec{p})$, R(s), and $h(\vec{k},\vec{z},0)$ appearing in equation (191) are all known. This equation is therefore a Volterra integral equation of the second kind. Such equations are very well suited to machine computation.

IV. NUMERICAL COMPUTATION OF ζ

Consider now a general Volterra integral equation of the second kind.

$$\zeta(s) - \int_0^s K(s,s')\,\zeta(s')\,ds' = \eta(s) \tag{192}$$

Here K(s,s') and $\eta(s)$ are known functions and are assumed bounded over the range of values of s which are of interest. We wish to determine $\zeta(s)$ for $0 \le s \le s_{max}$.

Let us divide this domain into N discrete increments

of length Δ. Let

$$\begin{aligned} s_0 &= 0 \\ &\cdots \\ s_n &= n\Delta \\ &\cdots \\ s_N &= s_{MAX} \end{aligned} \tag{193}$$

And let

$$\begin{aligned} \zeta(s_n) &= \zeta_n \\ \eta(s_n) &= \eta_n \\ K(s_n, s_{n'}) &= K_{n,n'} \end{aligned} \tag{194}$$

Evaluating equation (192) at s=0 we have

$$\zeta_0 = \eta_0 \tag{195}$$

We now convert equation (192) to an algebraic equation by approximating the integral by means of the trapezoidal rule.

$$\zeta_n - \sum_{n'=0}^{n} A_{n'} K_{n,n'} \zeta_{n'} \Delta = \eta_n \qquad n>0 \tag{196}$$

where

$$\begin{aligned} A_n &= \tfrac{1}{2} \qquad n'=0,n \\ A_n &= 1 \qquad 1 \le n' \le n-1 \end{aligned} \tag{197}$$

The ζ_n may now be determined by a simple iterative proceedure.

$$\zeta_n = \frac{\eta_n + \sum_{n'=0}^{n-1} A_{n'} K_{n,n'} \zeta_{n'} \Delta}{1 - \frac{1}{2} K_{n,n} \Delta} \tag{198}$$

Proceeding now to the solution of equation (191) let us choose the following initial two-particle correlation function.

$$\bar{g}^{(2)}(\vec{x}, \vec{z}_1, \vec{z}_2, 0) = \phi(\vec{z}_1)\phi(\vec{z}_2) S(\vec{x}, 0) \tag{199}$$

Here $\phi(\vec{z})$ is the single-particle velocity distribution function and $S(\vec{k},0)$ is the initial clustering spectrum. Comparing equations (199) and (180) we see that in the present case

$$f(\vec{x}, \vec{z}, 0) = \phi(\vec{z}) \tag{200}$$

and therefore

$$\int e^{-2\pi i \vec{x}\cdot\vec{s}} f(\vec{x}, \vec{z}, 0)\, d\vec{z} = \bar{\phi}(\vec{x}s) \tag{201}$$

Note too that in the present case equation (187) becomes

$$S(\vec{x}, s) = S(\vec{x}, 0) \left| \zeta(\vec{x}, s) \right|^2 \tag{202}$$

We shall choose $\phi(\vec{z})$ to be a normalized Gaussian.

Since the Fourier transform of a Gaussian of zero mean is another Gaussian of zero mean $\bar{\phi}(\vec{p})$ is of the form

$$\bar{\phi}(\vec{p}) = a\, e^{-bp^2} \tag{203}$$

The constants a and b are easily evaluated.

$$\begin{aligned} 1 &= \int \phi(\vec{z})\, d\vec{z} \\ &= \bar{\phi}(0) \\ &= a \end{aligned} \tag{204}$$

And

$$\begin{aligned} \langle z^2 \rangle &= \int \phi(\vec{z})\, z^2\, d\vec{z} \\ &= -\left(\frac{1}{2\pi}\right)^2 \left[\nabla_p^2 \int e^{-2\pi i \vec{p}\cdot\vec{z}} \phi(\vec{z})\, d\vec{z} \right]_{p=0} \\ &= -\frac{1}{(2\pi)^2} \left[\nabla_p^2\, \bar{\phi}(\vec{p}) \right]_{p=0} \\ &= \frac{6}{(2\pi)^2}\, b \end{aligned} \tag{205}$$

Thus

$$\bar{\phi}(\vec{p}) = e^{-\frac{(2\pi)^2 \langle z^2 \rangle\, p^2}{6}} \tag{206}$$

For simplicity the expansion parameter R(s) will be taken to correspond to the marginally bound universe. Setting $\alpha = 0$ in equation (31) we have

$$R(s) = \frac{6}{(\sqrt{6} - \omega s)^2} \tag{207}$$

It is convenient to use units of time and length such that

$$\omega^2 = 1 \tag{208}$$

and

$$\frac{(\text{dim}) \langle z^2 \rangle}{3} = 1 \tag{209}$$

In these units

$$\begin{aligned} n_n &= \bar{\phi}(\beta s_n) \\ &= e^{-\frac{1}{2}\beta^2 (n\Delta)^2} \end{aligned} \tag{210}$$

and

$$\begin{aligned} K_{n,n'} &= R(s_{n'})(s_n - s_{n'})\bar{\phi}[\beta(s_n - s_{n'})] \\ &= \frac{6}{(n'\Delta - \sqrt{6})^2} \Delta(n - n') e^{-\frac{1}{2}\beta^2[(n-n')\Delta]^2} \end{aligned} \tag{211}$$

Equations (210) and (211) may now be substituted into equation (198) to give $\bar{S}$ as a function of k and s.

In order for the results to be easily interpreted it is necessary to express them in terms of the physical

time t. Let us therefore integrate equation (15) using the form of the expansion parameter given in equation (207). We shall take the origin of time at the start of the expansion, corresponding to $s=-\infty$. Thus

$$t = \int_{-\infty}^{s} \left[\frac{6}{(\sqrt{6}-s')}\right]^3 ds' = \frac{12}{(\sqrt{6}-s)^3} \qquad (212)$$

The actual numerical computation of $\mathcal{S}$ was carried out with the aid of the I.B.M. 1620 computer at Brandeis University. The value of s_{max} was taken as 2.205 in order to allow R to increase by two full orders of magnitude. The number of increments N was taken as 100. The wave number k was allowed to take on 11 values in steps of 0.5 from 0.0 to 5.0. As a check on the accuracy obtained the calculation was repeated at 200 increments for k=0.0, 1.0, and 2.0 . These more accurate results differed from the 100 increment results by less than .5%. The final results are presented graphically on the following two pages and in tabular form in appendix C.

With the aid of equation (202) one may draw some simple conclusions with regard to the clustering spectrum from this data. It is clear that the portions of the spectrum

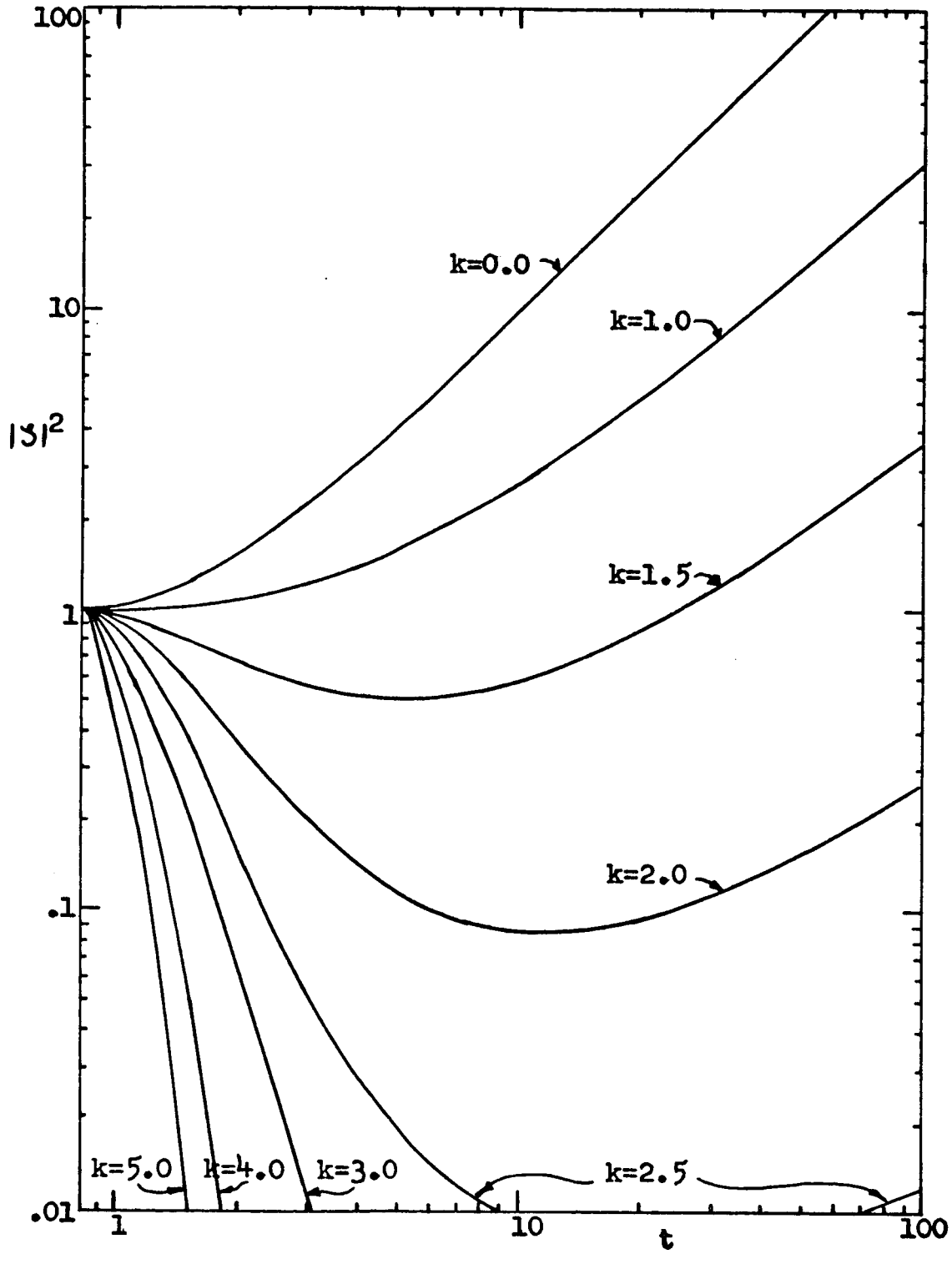


FIGURE 3

THE FUNCTION $|S(k,t)|^2$

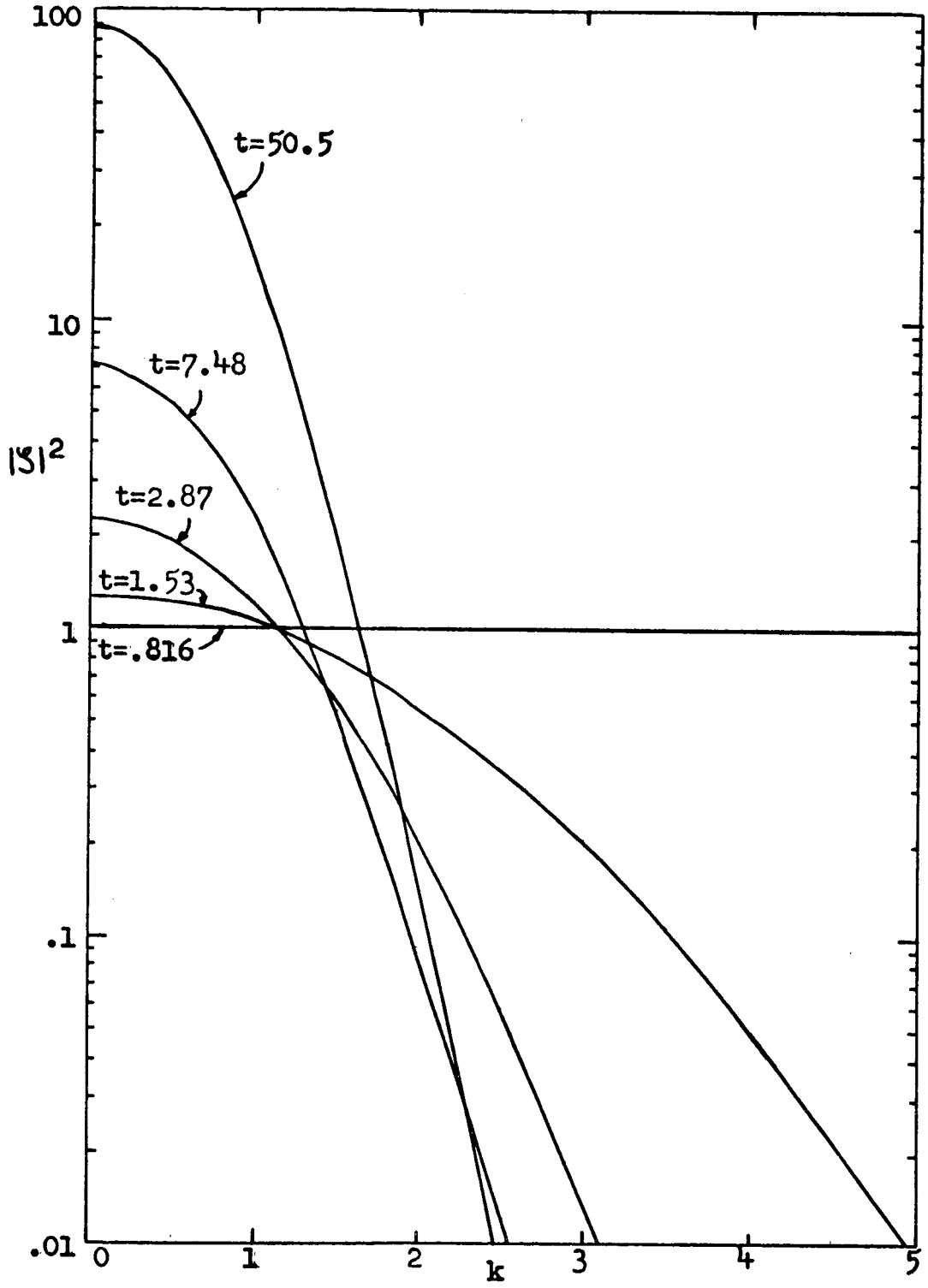


FIGURE 4

THE FUNCTION $|S(k,t)|^2$

at large wave numbers and therefore corresponding to short wave lengths tend to rapidly diminish. This is due to the phenomenon of Landau damping or phase mixing. The small wave number portion of the spectrum tends to rapidly grow with time. This is a reflection of the long-range nature of the gravitational force. We conclude that there is a rather poorly defined characteristic length Λ such that clustering on a smaller scale tends to diminish and clustering on a larger scale to grow. Somewhat arbitrarily taking the dimensionless critical wave number to be unity we may express Λ in terms of the physical quantities entering the problem. Refering to equations (208) and (209) we find

$$\Lambda^2 \approx \frac{1}{\omega^2}\frac{(2\pi)^2\langle z^2\rangle}{3} = \frac{\langle z^2\rangle}{\bar{\rho}(0)\,G}\,\frac{\pi}{3} \tag{213}$$

This is substantially the result of Jeans (10). Of course the details of the time development of the irregularities as developed in this thesis differ substantially from that early work. In particular Jeans took for the unperturbed system a static universe, now known to be contradicted by observation. Also, Jeans chose to represent the material of the universe as a compressible fluid. This real fluid differs from what has been called in this thesis the continuum limit of a distribution of particles by possessing at each

point a single velocity of flow as well as a certain pressure. As a consequence, Jeans found that small wave length disturbances propagated as sound waves rather than simply being dissipated as has been shown here.

A more relevant treatment has been given by Lifshitz (12). In this general relativistic analysis the unperturbed state was taken to be an Einstein-Friedmann expanding universe. Here too however, the material was assumed to be an ordinary fluid and so again no Landau damping was found.

V. THE EFFECT OF NON-LINEARITY

It was mentioned in section II. that interaction between different parts of the clustering spectrum results from the occurence of non-linear terms in equations (161). Let us see how this comes about.

Differentiating the second of equations (161) with respect to s on obtains, among others, a contribution to $\frac{\partial^2 g^{(2)}}{\partial s^2}$ of the form

$$C \sum_{j=1}^{2} \int \vec{F}_{j,3} \cdot \frac{\partial}{\partial \vec{z}_j} \frac{\partial g^{(3)}(1,2,3)}{\partial s} \, d(3)$$

Substituting for the time derivative of $g^{(3)}$ the quantity obtained from the third of these equations one discovers a partial contribution to $\frac{\partial^2 g^{(2)}}{\partial s^2}$ of the form

$$c^{2}\int\left(\vec{F}_{1,3}\cdot\frac{\partial}{\partial\vec{z}_{1}}+\vec{F}_{2,3}\cdot\frac{\partial}{\partial\vec{z}_{2}}\right)\left\{\vec{F}_{1,4}\cdot\frac{\partial}{\partial\vec{z}_{1}}\left[g^{(2)}(1,2)g^{(2)}(3,4)+g^{(2)}(1,3)g^{(2)}(2,4)\right]\right.$$

$$\left.+\vec{F}_{2,4}\cdot\frac{\partial}{\partial\vec{z}_{2}}\left[g^{(2)}(1,2)g^{(2)}(3,4)+g^{(2)}(2,3)g^{(2)}(1,4)\right]\right\}d(3)d(4)$$

Since $g^{(2)}$ appears quadratically this expression will, upon Fourier transformation, result in the occurence of a convolution integral of the form

$$\int\psi(\vec{k}')\chi(\vec{k}-\vec{k}')\,d\vec{k}'$$

Such integrals represent, in some sense, the interaction of different parts of the clustering spectrum.

This property of non-linear terms resulting in the coupling of different parts of a spectrum function is actually quite general. In the theory of hydrodynamic turbulence, for example, the quadratic appearance of the velocity in the convective or inertial term $\vec{v}\cdot\nabla\vec{v}$ results in a similar transfer of energy between different parts of the corresponding kinetic energy spectrum.

CHAPTER VI

SUMMARY

At this point it may be useful to summarize the physical results of the work that has been presented.

First of all, we have seen that it is possible to introduce a set of coordinates in which the expanding system of gravitating particles assumes the form of a non-expanding distribution with a negative mass background and an explicit time dependence of the gravitational constant.

It was shown too that the correlation functions provide a convenient description of the statistical distribution of particles. In particular, the discussion of hierarchical clustering showed that the vanishing of these functions corresponds to the absence of correlations in the intuitive sense.

The two-particle correlation function was used to redefine Layzer's peculiar binding energy in order to obtain a quantity more appropriate to a particulate distribution. It was shown that this redefined quantity obeys precisely the same energy equation as Layzer's original quantity.

APPENDIX A

FUNCTIONAL DIFFERENTIATION

Functional differentiation plays the same role in the theory of functionals as partial differentiation does in the theory of functions of multiple variables. The operation was first introduced by V. Volterra (11).

Let $\vec{x}$ be a set of variables in some space. For the applications of this thesis $\vec{x}$ will denote a point in six-dimensional position-velocity space. Let $\Psi[w]$ be a functional of an arbitrary function of $\vec{x}$, $w(\vec{x})$. The functional derivative of Ψ with respect to w at the point $\vec{x}$ is defined to be

$$\frac{\delta \Psi[w]}{\delta w(\vec{x})} = \lim_{\substack{\epsilon \to 0 \\ \eta \to 0}} \frac{\Psi[w + \Theta_\epsilon] - \Psi[w]}{\int \Theta_\epsilon(\vec{x}')\, d\vec{x}'} \qquad \text{(A 1a)}$$

where Θ_ϵ is any function satisfying the conditions

$$\Theta_\epsilon(\vec{x}') = 0 \qquad |\vec{x} - \vec{x}'| \geq \eta \qquad \text{(A 1b)}$$

and

$$0 \leq \Theta_\epsilon(\vec{x}') \leq \epsilon \qquad |\vec{x} - \vec{x}'| < \eta \qquad \text{(A 1c)}$$

The functional derivative of Ψ with respect to w at the point $\vec{x}$ is thus the ratio of the increment in Ψ ,

The actual calculation of the clustering spectrum for a three-level hierarchical distribution showed that the occurence of clustering on some well defined linear scale, say L, does not necessarily imply a peak in the clustering spectrum at k=1/L. It appears that the stability of the clustering spectrum in the ordinary sense is not quite equivalent to the stability of the physical clustering hierarchy.

In discussing the general structure of the equations of motion for the correlation functions it was seen that the interesting problem of the transfer of energy between different parts of the clustering spectrum requires a non-linear analysis which has not been attempted.

Finally, the numerical calculation of the development of the clustering spectrum in the linear approximation showed that large-scale irregularities grow because of gravitational clustering while small-scale irregularities dissipate as a result of Landau damping. The dividing line between these two types of behavior is approximately the Jeans critical wave length.

when its argument function is varied very slightly in the neighborhood of $\vec{x}$, to the integral of this variation.

The most basic class of functionals are the monomials. Let $f(\vec{x}_1 \ldots \vec{x}_n)$ be a definite function of n points in $\vec{x}$ space. f generates a monomial functional of the n'th degree.

$$\Psi[w] = \int f(\vec{x}_1 \ldots \vec{x}_n) w(\vec{x}_1) \ldots w(\vec{x}_n) d\vec{x}_1 \ldots d\vec{x}_n \qquad \text{(A 2)}$$

We may, without any loss of generality, assume f to be symmetric under any interchange of its arguments. The functional derivative of Ψ is calculated by substituting equation (A 2) into equations (A 1) and dropping terms of higher order than the first in θ (and therefore in ϵ). The result is

$$\frac{\delta}{\delta w(\vec{x})} \int f(\vec{x}_1 \ldots \vec{x}_n) w(\vec{x}_1) \ldots w(\vec{x}_n) d\vec{x}_1 \ldots d\vec{x}_n = n \int f(x, x_2 \ldots x_n) w(\vec{x}_2) \ldots w(\vec{x}_n) d\vec{x}_2 \ldots d\vec{x}_n \qquad \text{(A 3)}$$

Notice that the result is a monomial of one degree less. This process may be repeated. In particular the n-fold derivative is given by

$$\frac{\delta^n}{\delta w(\vec{x}_1') \ldots \delta w(\vec{x}_n')} \int f(\vec{x}_1 \ldots \vec{x}_n) w(\vec{x}_1) \ldots w(\vec{x}_n) d\vec{x}_1 \ldots d\vec{x}_n = n!\, f(\vec{x}_1' \ldots \vec{x}_n') \qquad \text{(A 4)}$$

The function $w(\vec{x})$ may itself be regarded as a monomial functional of the first degree by writing it as

$$w(\vec{x}) = \int \delta(\vec{x} - \vec{x}')\, w(\vec{x}')\, d\vec{x}' \qquad \text{(A 5)}$$

It therefore follows that the functional derivative of $w(\vec{x})$ is given by

$$\frac{\delta w(\vec{x})}{\delta w(\vec{x}')} = \delta(\vec{x} - \vec{x}') \qquad \text{(A 6)}$$

Note the similarity to the usual result for partial differentiation with respect to the independent variables $w_1 .. w_n$.

$$\frac{\partial w_i}{\partial w_j} = \delta_{ij} \qquad \text{(A 7)}$$

The usual rules of differentiation hold for functional differentiation as well. Thus, if Φ is a function of a single argument, and if Φ' is its derivative with respect to this argument, the chain rule holds in the following form.

$$\frac{\delta \Phi(\Psi[w])}{\delta w(\vec{x})} = \Phi' \frac{\delta \Psi[w]}{\delta w(\vec{x})} \qquad \text{(A 8)}$$

The usual rule for the derivative of a product holds too.

$$\frac{\delta}{\delta w(\vec{x})} \left\{ \Phi[w]\, \Psi[w] \right\} = \Phi \frac{\delta \Psi}{\delta w(\vec{x})} + \Psi \frac{\delta \Phi}{\delta w(\vec{x})} \qquad \text{(A 9)}$$

Finally, functional differentiation commutes with

integration over $\vec{x}$ in the same way that partial differentiation with respect to w_i commutes with summation over the index.

APPENDIX B

THE JOINT-GAUSSIAN RANDOM FIELD

A collection of N random variables $x_1 .. x_N$ is said to be distributed according to a joint-Gaussian law if the probability density for the distribution is of the form

$$P(x_1 .. x_N) = \text{Const.} \times e^{-\frac{1}{2}\sum_{j,k=1}^{N} a_{jk}(x_j - \langle x_j \rangle)(x_k - \langle x_k \rangle)} \tag{B 1}$$

where the constant is chosen so that P integrates to unity.

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(x_1 .. x_N)\, dx_1 .. dx_N = 1 \tag{B 2}$$

The matrix a_{jk} may be taken to be symmetric.

Associated with this probability density there is a characteristic function $\phi(y_1 .. y_N)$ from which the various moments of the distribution are obtained by differentiation.

$$\begin{aligned} \phi(y_1 .. y_N) &\equiv \left\langle e^{i\sum_{j=1}^{N} x_j y_j} \right\rangle \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \text{Const.} \cdot e^{-\frac{1}{2}\sum_{j,k} a_{jk}(x_j - \langle x_j \rangle)(x_k - \langle x_k \rangle) + i\sum_j x_j y_j}\, dx_1 .. dx_N \end{aligned} \tag{B 3}$$

And

$$\langle x_{j_1} x_{j_2} \cdots x_{j_n} \rangle = \left(\frac{1}{i}\right)^n \left[\frac{\partial^n \phi}{\partial y_{j_1} \cdots \partial y_{j_n}} \right]_{\{y=0\}} \tag{B 4}$$

In order to perform the integration required in

equation (B 3) we assume that the matrix a is non-singular. Let a^{-1} be the inverse of a and det a its determinant. We may then write the exponent appearing in (B 3) as follows.

$$-\frac{1}{2}\sum_{j,k=1}^{N} a_{jk}(x_j - \langle x_j\rangle)(x_k - \langle x_k\rangle) + i\sum_{j=1}^{N} x_j Y_j$$

$$= -\frac{1}{2}\sum_{j,k=1}^{N} a^{-1}_{jk}\Big[\sum_{\ell=1}^{N} a_{j\ell}(x_\ell - \langle x_\ell\rangle) - iY_j\Big]\Big[\sum_{m=1}^{N} a_{km}(x_m - \langle x_m\rangle) - iY_k\Big] \qquad \text{(B 5)}$$

$$+ i\sum_{j=1}^{N}\langle x_j\rangle Y_j - \frac{1}{2}\sum_{j,k=1}^{N} a^{-1}_{jk} Y_j Y_k$$

We introduce a new set of variables given by

$$\bar{x}_j = \sum_{\ell} a_{j\ell}(x_\ell - \langle x_\ell\rangle) - iY_j \qquad \text{(B 6)}$$

The Jacobian of this transformation is

$$\frac{D(x_1 \ldots x_N)}{D(\bar{x}_1 \ldots \bar{x}_N)} = \frac{1}{\det a} \qquad \text{(B 7)}$$

Substituting the new variables into equation (B 3) with the aid of equation (B 5) we find

$$\phi(Y_1 \ldots Y_n) = e^{i\sum Y_j\langle x_j\rangle - \frac{1}{2}\sum a^{-1}_{jk} Y_j Y_k}$$

$$\times \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{\text{Const}}{\det a}\, e^{-\frac{1}{2}\sum a^{-1}_{jk}\bar{x}_j\bar{x}_k}\, d\bar{x}_1 \ldots d\bar{x}_N \qquad \text{(B 8)}$$

The remaining integral no longer depends on the y's. It is thus simply a number and is evaluated merely by requiring

$$\phi(0,0,\ldots,0) = 1 \qquad \text{(B 9)}$$

which follows from the normalization of the probability

density P. The characteristic function ϕ is then given by

$$\phi(\gamma_1 \dots \gamma_N) = e^{i\sum_j \langle x_j \rangle \gamma_j - \frac{1}{2}\sum_{j,k} a^{-1}_{jk} \gamma_j \gamma_k} \tag{B 10}$$

The matrix a^{-1} may be evaluated in terms of the second moments of the distribution by differentiating the characteristic function twice and setting the y's to zero.

$$\left.\frac{\partial^2 \phi}{\partial \gamma_j \partial \gamma_k}\right]_{\{\gamma=0\}} = -\langle x_j x_k \rangle = -\langle x_j \rangle \langle x_k \rangle - a^{-1}_{jk} \tag{B 11}$$

Thus

$$a^{-1}_{jk} = \langle x_j x_k \rangle - \langle x_j \rangle \langle x_k \rangle = \langle (x_j - \langle x_j \rangle)(x_k - \langle x_k \rangle) \rangle \tag{B 12}$$

The characteristic function is therefore

$$\phi(\gamma_1 \dots \gamma_N) = e^{i\sum_j \langle x_j \rangle \gamma_j - \frac{1}{2}\sum_{j,k} \langle (x_j - \langle x_j \rangle)(x_k - \langle x_k \rangle) \rangle \gamma_j \gamma_k} \tag{B 13}$$

A random field $p(\vec{y},\vec{z})$ may be said to be distributed according to a joint-Gaussian law if the values of p at any N points in position-velocity space are joint-Gaussian. Let us consider $p(\vec{y},\vec{z})$ to be the limit of a sequence of functions which are piecewise continuous over small regions in posi-

tion-velocity space denoted by Δ_j. Let $p(j)$ be the constant value of $p(\vec{y},\vec{z})$ in cell j. The characteristic functional of the joint-Gaussian random field p is then given by

$$\Phi[w] \equiv \left\langle e^{i\int p(\vec{y},\vec{z})\, w(\vec{y},\vec{z})\, d\vec{y}\, d\vec{z}} \right\rangle$$

$$= \lim_{\{\Delta_j \to 0\}} \left\langle e^{i \sum_j p(j) \int_{\Delta_j} w(\vec{y},\vec{z})\, d\vec{y}\, d\vec{z}} \right\rangle$$

$$= \lim_{\{\Delta_j \to 0\}} \phi\left(\int_{\Delta} w(\vec{y},\vec{z})\, d\vec{y}\, d\vec{z}, \ldots, \int_{\Delta_N} w(\vec{y},\vec{z})\, d\vec{y}\, d\vec{z} \right)$$

$$= \lim_{\{\Delta_j \to 0\}} e^{i \sum_j \langle p(j) \rangle \int_{\Delta_j} w(\vec{y},\vec{z})\, d\vec{y}\, d\vec{z} - \frac{1}{2} \sum_{j,k} \langle \tilde{p}(j)\, \tilde{p}(k) \rangle \int_{\Delta_j} w(\vec{y},\vec{z})\, d\vec{y}\, d\vec{z} \int_{\Delta_k} w(\vec{y},\vec{z})\, d\vec{y}\, d\vec{z}} \qquad \text{(B 14)}$$

$$= e^{i \int \langle p(\vec{y},\vec{z}) \rangle w(\vec{y},\vec{z})\, d\vec{y}\, d\vec{z} - \frac{1}{2} \int\int \langle \tilde{p}(\vec{y},\vec{z})\, \tilde{p}(\vec{y}',\vec{z}') \rangle w(\vec{y},\vec{z})\, w(\vec{y}',\vec{z}')\, d\vec{y}\, d\vec{z}\, d\vec{y}'\, d\vec{z}'}$$

$$= e^{i \int \bar{p}(1) w(1)\, d(1) - \frac{1}{2} \int\int \langle \tilde{p}(1)\, \tilde{p}(2) \rangle w(1)\, w(2)\, d(1)\, d(2)}$$

APPENDIX C

NUMERICAL RESULTS

On the following three pages appear some of the numerical results of the computation of $\mathcal{S}(k,t)$ as described in chapter V. Since it is $|\mathcal{S}|^2$ which indicates the growth of the clustering spectrum it is this function which is tabulated rather than $\mathcal{S}$. For the sake of completeness R(t) is tabulated too. As mentioned in the text the following data correspond to a division of the range of the time-like variable s into one hundred equal increments.

The origin of time has been chosen to be the start of the expansion. The data is tabulated in terms of dimensionless variables

$$\bar{t} = t\sqrt{4\pi \bar{\rho}(0) G} \qquad (C\ 1)$$

and

$$\bar{k} = k\sqrt{\frac{\langle z^2 \rangle \pi}{\bar{\rho}(0) G\, 3}} \qquad (C\ 2)$$

where $\bar{\rho}(0)$ is the average density of matter at the initial time, (the time at which R=1), and $\langle z^2 \rangle$ is the average square of the peculiar velocity at that time.

$\bar{t}$	$R(\bar{t})$	$\lvert\mathcal{S}(\bar{k},\bar{t})\rvert^2$		
		$\bar{k}=0.0$	$\bar{k}=0.5$	$\bar{k}=1.0$
8.16×10^{-1}	$1.00\times10^{+0}$	$1.00\times10^{+0}$	$1.00\times10^{+0}$	$1.00\times10^{+0}$
$1.15\times10^{+0}$	1.26	1.08	1.06	1.01
1.53	1.52	1.27	1.20	1.03
2.10	1.88	1.64	1.44	1.11
2.87	2.31	2.25	1.86	1.20
4.06	2.91	3.34	2.58	1.41
5.43	3.53	4.75	3.48	1.66
7.48	4.38	7.13	4.93	2.07
$1.01\times10^{+1}$	5.34	$1.05\times10^{+1}$	6.90	2.62
1.40	6.65	1.61	$1.01\times10^{+1}$	3.48
1.87	8.08	2.36	1.43	4.56
2.59	$1.00\times10^{+1}$	3.63	2.12	6.27
3.39	1.20	5.18	2.93	8.27
5.05	1.56	8.80	4.79	$1.27\times10^{+1}$
7.09	1.96	$1.38\times10^{+2}$	7.27	1.84
9.09	2.32	1.93	9.89	2.44
$1.19\times10^{+2}$	2.78	2.76	$1.38\times10^{+2}$	3.34
1.61	3.39	4.11	2.00	4.72
2.24	4.23	6.40	3.02	6.96
3.26	5.42	$1.05\times10^{+3}$	4.79	$1.08\times10^{+2}$
4.00	6.22	1.38	6.16	1.38
6.34	8.45	2.54	$1.48\times10^{+3}$	3.22
8.21	$1.00\times10^{+2}$	3.59	2.08	4.50

$\bar{t}$	$\lvert\vartheta(\bar{k},\bar{t})\rvert^2$			
	$\bar{k}=1.5$	$\bar{k}=2.0$	$\bar{k}=2.5$	$\bar{k}=3.0$
8.16×10^{-1}	$1.00\times10^{+0}$	$1.00\times10^{+0}$	$1.00\times10^{+0}$	$1.00\times10^{+0}$
$1.15\times10^{+0}$	9.22×10^{-1}	8.43×10^{-1}	6.99×10^{-1}	5.79×10^{-1}
1.53	7.97	5.86	3.52	2.01
2.10	6.67	3.55	1.40	4.92×10^{-2}
2.87	5.75	2.19	5.85×10^{-2}	1.36
4.06	5.22	1.42	2.70	4.46×10^{-3}
5.43	5.13	1.10	1.68	2.20
7.48	5.38	9.25×10^{-2}	1.16	1.19
$1.01\times10^{+1}$	5.94	8.58	9.10×10^{-3}	7.68×10^{-4}
1.40	6.98	8.52	7.71	5.35
1.87	8.37	8.97	7.18	4.29
2.59	$1.06\times10^{+0}$	9.99	7.14	3.71
3.39	1.32	1.13×10^{-1}	7.47	3.51
5.05	1.90	1.44	8.62	3.56
7.09	2.66	1.83	1.03×10^{-2}	3.90
9.09	3.44	2.23	1.21	4.33
$1.19\times10^{+2}$	4.61	2.80	1.48	5.00
1.61	6.44	3.68	1.89	6.07
2.24	9.44	5.05	2.56	7.79
3.26	$1.47\times10^{+1}$	7.34	3.69	1.07×10^{-3}
4.00	1.88	9.07	4.56	1.30
6.34	3.31	$1.95\times10^{+0}$	7.48	2.04
8.21	4.57	2.65	1.00×10^{-1}	2.68

$\bar{t}$	$\lvert S(\bar{k},\bar{t})\rvert^2$			
	$\bar{k}=3.5$	$\bar{k}=4.0$	$\bar{k}=4.5$	$\bar{k}=5.0$
8.16×10^{-1}	$1.00\times10^{+0}$	$1.00\times10^{+0}$	$1.00\times10^{+0}$	$1.00\times10^{+0}$
$1.15\times10^{+0}$	4.62×10^{-1}	3.57×10^{-1}	2.66×10^{-1}	1.92×10^{-1}
1.53	1.04×10^{-1}	4.93×10^{-2}	2.13×10^{-2}	8.45×10^{-3}
2.10	1.49×10^{-2}	4.01×10^{-3}	9.80×10^{-4}	2.25×10^{-4}
2.87	2.75×10^{-3}	5.04×10^{-4}	8.66×10^{-5}	1.40×10^{-5}
4.06	6.46×10^{-4}	8.37×10^{-5}	9.71×10^{-6}	1.01×10^{-6}
5.43	2.46	2.39	2.02	1.53×10^{-7}
7.48	1.02	7.41×10^{-6}	4.70×10^{-7}	2.67×10^{-8}
$1.01\times10^{+1}$	5.25×10^{-5}	3.07	1.56	7.01×10^{-9}
1.40	3.00	1.41	5.75×10^{-8}	2.05
1.87	2.06	8.27×10^{-7}	2.83	8.43×10^{-10}
2.59	1.53	5.24	1.52	3.81
3.39	1.30	3.98	1.02	2.26
5.05	1.15	3.04	6.70×10^{-9}	1.26
7.09	1.14	2.72	5.37	9.03×10^{-11}
9.09	1.19	2.65	4.88	7.63
1.19×10^{2}	1.30	2.71	4.66	6.78
1.61	1.49	2.92	4.70	6.40
2.24	1.81	3.35	5.07	6.47
3.26	2.37	4.14	5.92	7.09
4.00	2.79	4.76	6.62	7.70
6.34	4.21	6.82	8.99	9.90
8.21	5.41	8.57	1.10×10^{-8}	1.18×10^{-10}

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